University of Al-Hamdaniya, College of Education
Department of Mathematics

GROUP THEORY

Level Two

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Lecture No. 2

Definition: A semi group is a pair (S,*) consisting of a nonempty sets together with an associative binary operation * defined on S.

Example: Let Q be the rational numbers, define $a * b = \frac{1}{2}(a+b): a, b \in Q$. prove if (Q,*) is a semi group or not?

Solution: $a * b = \frac{1}{2}(a+b): a, b \in Q$ then $a * b \in Q$

$$\therefore * \text{ is closed}$$

let $a, b, c \in Q$

$$a * (b * c) = (a * b) * c$$

L.S./ $a * (b * c) = a * \left[\frac{1}{2}(b+c)\right]$

$$= \frac{1}{2}\left[a + \left[\frac{1}{2}(b+c)\right] = \frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}c\right.$$

R.S./ $(a * b) * c = \frac{1}{2}(a + b) * c = \frac{1}{2}\left[\frac{1}{2}(a + b) + c\right]$

$$= \frac{1}{4}a + \frac{1}{4}b + \frac{1}{2}c$$

$$\because \text{L. S.} \neq \text{R. S.}$$

∴ * is not associative

Then (Q,*) is not semi group.

<u>Definition</u>: The system (S,*) is said to have a (two- sides) identity element for the operation * if there exists an element e in S such that:

$$a * e = e * a = a \text{ for every } a \in S$$

<u>Example</u>: (0) is the identity element for the systems (Z,+),(Q,+),(R,+) and (1) for (N,.),(Z,.),(Q,.),(R,.).

$(Z_e,.)$ has not identity element

Example2: Let $S = \{a + b\sqrt{2} : \quad a, b \in Z\}$ is the system $(S, .)$ has an identity element?

Sol. $\forall\ a + b\sqrt{2} \in S \quad \exists\ e_1 + e_2\sqrt{2} \in S$, s.t.

$$(a + b\sqrt{2})(e_1 + e_2\sqrt{2}) = (e_1 + e_2\sqrt{2})(a + b\sqrt{2}) = (a + b\sqrt{2})$$

L.S. / $(a + b\sqrt{2})(e_1 + e_2\sqrt{2}) = a + b\sqrt{2}$

$$ae_1 + 2be_2 + (ae_2 + be_1)\sqrt{2} = a + b\sqrt{2}$$

$$ae_1 + 2be_2 = a \ldots \quad (1)$$

$$\rightarrow e_1 = \frac{a - 2be_2}{a} \quad \ldots (3)$$

$$ae_2 + be_1 = b \ \ldots \quad (2)$$

Substitute 3 in 2 we get

$$ae_2 + \frac{ba - 2b^2 e_2}{a} = b$$

$$a^2 e_2 + ba - 2\,b^2 e_2 - ba = 0$$

$$(a^2 - 2\,b^2)e_2 = 0$$

$$\rightarrow e_2 = 0$$

$$\rightarrow e_1 = 1$$

$$\therefore\; e_1 + e_2\sqrt{2} = 1 + 0\sqrt{2}$$

R.S./ Similar