University of Al-Hamdaniya, College of Education
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GROUP THEORY

Level Two

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Lecture No. 5

Definition: the operation * defined on the set S is called commutative if a*b=b*a for every pair of elements $a, b \in S$.

Example1: Let S=Z , a*b=a+b-1

Solution: let $a, b \in Z$ S.T:

$$a*b=a+b-1=b+a-1=b*a$$

∴* is a commutative.

Example2: Let S=R/{0} , $a*b = \frac{a}{b}$ then * is not comm. operation.

Definition: Let (G,*) be a group, if * is a commutative operation on G then (G,*) is called a commutative group.

Example: (Z,+), (R,+), (Q,+), (Q/{0},.),(R/{0},.) are comm. group.

Theorem1: Let (G,*) be a group , $a \in G$ and $m, n \in Z$. the powers of a obey the following laws of exponents:

1. $a^n * a^m = a^{n+m} = a^m * a^n$

2. $(a^n)^m = a^{nm} = (a^m)^n$

3. $a^{-n} = (a^n)^{-1}$

4. $e^n = e$

Theorem2: The identity element of a group (G,*) is unique, and each element of a group has inverse element.

Proof: Let (G,*) be a group has two identity element e_1 and e_2 then $\forall\, a \in G$:

$a * e_1 = e_1 * a = a \;\ldots(1)$

and

$a * e_2 = e_2 * a = a \;\ldots(2)$

Equality (1) and (2) we have

$$a * e_1 = a * e_2 \rightarrow e_1 = e_2$$

So, the identity element of a group (G,*) is unique.

To show that an element of a group (G,*) has exactly one inverse, we assume that $a \in G$ such that a has two inverse element a_1^{-1} *and* a_2^{-1} then

$$a * a_1^{-1} = a_1^{-1} * a = e \dots (3)$$

$$a * a_2^{-1} = a_2^{-1} * a = e \dots (4)$$

Equality (3) and (4) we have

$$a * a_1^{-1} = a * a_2^{-1} \dots . (5)$$

Multiply both sides of (5) from the left by a_1^{-1} or a_2^{-1} we have

$$(a_1^{-1} * a) * a_1^{-1} = (a_1^{-1} * a) * a_2^{-1}$$

$$\rightarrow e * a_1^{-1} = e * a_2^{-1}$$

$$\rightarrow a_1^{-1} = a_2^{-1}$$

∴ each element of a group (G,*) has exactly one inverse element