University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 10

Definition : let R be a ring and I is an ideal of R. Then $a + I = \{a + i : i \in I, a \in R\}$ is coset of I in the ring R.

Notes :

1) $(a + I) \oplus (b + I) = (a + b) + I$

2) $(a + I) \otimes (b + I) = (ab) + I$

3) $R/I = \{ a + I : a \in R\}$

4) $a + I = b + I \leftrightarrow a - b \in I$

5) $a + I = I$ iff $a \in I$

<u>Example1:</u> (Z_6,+,.) is a ring ,(2) = {0,2,4}   is an ideal of (Z6,+,.) is an ideal of (Z6,+,.) then

$$Z_6/(2) = \{0 + (2), 1 + (2)\}$$

<u>Example</u> 2: (Z,+.)is a ring (Z_e,+.) is an ideal of (Z,+,.) then

$$Z/Z_e = \{0 + Z_e, 1 + Z_e\} = \{Z_e, Z_o\}$$

Theorem5: If I is an ideal of the ring (R,+,.) ,then (R/I,⊕,⊗) is a ring which is called (Quotient ring of R by I)

proof: I) (R/I,⊕) is a comm. group

1) $(a + I) \oplus (b + I) = (a + b) + I \in R/I$

$$\forall a + I, b + I \in R/I$$

1) $\text{Let } a + I, b + I, c + I \in R/I$

$$[(a + I) \oplus (b + I)] \oplus (c + I) = (a + I) \oplus [(b + I) \oplus (c + I)]$$

$$L.S.\ [(a + I) \oplus (b + I)] \oplus (c + I) = ((a + b) + I) \oplus (c + I)$$

$$= (a + (b + c) + I] = (a + I) \oplus [(b + c) + I]$$

$$= (a + I) \oplus [(b + I) \oplus (c + I)]\ R.S$$

∴ ⊕ is a associative

3) $\exists e + I \in R/I \quad \forall \; a + I \in R/I \quad S.T.$

H.W

4) for each $a + I \in R/I \;\; \exists a^{-1} + I \; \in R/I$ S.T.

H.W

5)$(a + I) \oplus (b + I) = (b + I) \oplus (a + I)$

L.S. $(a + I) \oplus (b + I) = (a + b) + I = (b + a) + I = (b + I) \oplus (a + I) = R.S$

$\therefore \oplus$ is comm.

II) $(R/I, \otimes)$ is a semi group

1) Let $a + I, b + I \in R/I$

$\Rightarrow (a + I) \otimes (b + I) = (ab) + I \in R/I$

2) Let $a + I, b + I, c + I \in R/I$

$[(a + I) \otimes (b + I)] \otimes (c + I) = (a + I) \otimes [(b + I) \otimes (c + I)]$ H.W

$\therefore \otimes$ is associative

III) $\otimes$ is dist. over

1) $(a + I) \otimes [(b + I) \oplus (c + I)] = [(a + I) \otimes (b + I)] \oplus [(a + I) \otimes (c + I)]$ H.W

2) By the same way (H.W)

$\therefore (R/I, \oplus, \otimes)$ is a ring

Theorem6: If the ring $(R,+,.)$ is commutative then the quotient ring $\therefore(R/I,\oplus,\otimes)$ is also.

Proof: let Let $a+I, b+I \in R/I$ then

$$(a+I)\otimes(b+I)=(ab)+I$$

$$=(ba)+I \quad [\text{since R is a comm. ring}]$$

$$=(b+I)\otimes(a+I)$$

$\therefore$ the ring $(R/I,\oplus,\otimes)$ is commutative.

Theorem7: If (R,+,.) is a ring with identity, so the ring (R/I,⊕,⊗) is with identity.

Proof: since a ring (R,+,.) is a ring with identity, then ∃ 1 ∈ R s.t.

$$a.1 = 1.a = a \;\forall\, a \in R$$

Now, let $\mathrm{a + I \in R/I}$ then $\mathrm{a + I = (a.1) + I = (a + I) \otimes (1 + I)}$

Similarly/ $\mathrm{a + I = (1.a) + I = (1 + I) \otimes (a + I)}$

∴ the ring (R/I,⊕,⊗)has an identity element.

Example: $(Z_{12},+,.)$ is comm. Ring with identity and $((2),+,.)$ is an ideal of Z_{12} also $(Z_{12}/(2),\oplus,\otimes)$is comm. ring with identity.