University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 15

Definition: Let (R,+,.) and $(R',+',.')$ are rings and let $f: R \to R'$ be a function then f is said to be a ring homomorphism iff:

1) $f(a+b)=f(a)+' f(b)$

2) $f(a.b)=f(a).' f(b)$

$\forall a, b \in R$.

Example(3): Let (R,+,.) be a ring with identity and $f_a : (R,+,.) \to (R,+,.)$

defined as:

$$f_a(x) = axa^{-1} \quad \forall x, a \in R$$

Show that f_a is a ring homomorphism.

Solution:1) $L.S)\ f_a(x+y) = a(x+y)a^{-1}$

$$= axa^{-1} + aya^{-1}$$

$$= f_a(x) + f_a(y) = R.S$$

2) $L.S)\ f_a(x.y) = a(x.y)a^{-1}$

$$= a(x.1.y)a^{-1}$$

$$= a(x\,a^{-1}ay)a^{-1}$$

$= (axa^{-1}).(aya^{-1})$

$= f_a(x).f_a(y) = R.S$

$\therefore f_a$ is a ring homomorphism.

Definition: Let f be a ring homomorphism from the ring $(R, +, .)$ into the ring $(R', +', .')$ then the kernel of f is defined by:

$$\ker(f) = \{a \in R : f(a) = 0'\}$$

Example: Let $f: (Z, +, .) \rightarrow (Z_2, +, .)$ defined by:

$$f(a) = \begin{cases} 0 & if \quad a \in Z_e \\ 1 & if \quad a \in Z_o \end{cases}$$

Find ker(f)?

Solution: we must prove that f is a ring homomorphism

I)Let $a, b \in Z_e$

$\Rightarrow a + b \in Z_e$, $a.b \in Z_e$

1)f(a+b) =0

=0+0 = f(a)+ f(b)

2) f(a.b) =0

=0.0 = f(a). f(b)

II) Let $a, b \in Z_o$

$\Rightarrow a + b \in Z_e$, $a.b \in Z_O$

1)L.S) f(a+b)=0

R.S) f(a)+ f(b)= 1+1=0

2)L.S) f(a.b)=1

R.S) f(a). f(b)= 1.1=1

III) $a \in \boldsymbol{Z_e}$, $b \in \boldsymbol{Z_o}$,

$\Rightarrow a + b \in Z_o$, $a.b \in Z_e$

1)L.S) f(a+b)=1

R.S) f(a)+ f(b)=0+1=1

2)L.S) f(a.b)=0

R.S) f(a). f(b)=0.1=0

$\therefore$ f is a ring homomorphism.

$$\ker(f) = \{a \in Z \quad : \quad f(a) = 0'\}$$

$$= \{a \in Z_e \quad : \quad f(a) = 0\}$$

$$\therefore \ker(f) = Z_e$$