University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 19

Theorem11: Let f be a ring homomorphism from $(R, +, .)$ into the ring $(R', +', .')$ then:

1) $f(0)=0'$

2) $f(-a)=-f(a) \ \forall a \in R$

3) $(f(R),+',.')$ is a subring of $(R',+',.')$.

If (R, +, .) and (R′,+′,.′) are rings with identity element 1&1′ then

4) f(1)=1′.

$proof$: $f(a) = f(a.1) = f(a).' f(1)$ [since f is homo.]

$f(a).' 1' = f(a).' f(1)$ $[by\ cacellation\ law]$

$\therefore f(1) = 1'.$

5) $f(a^{-1}) = (f(a))^{-1} \quad \forall a \in R$

Proof : Let $f(a) \in R'$

$f(a).' f(a^{-1}) = f(aa^{-1}) = f(e) = e'$

$\therefore f(a^{-1})$ is inverse $f(a)$

But $(f(a))^{-1}$ is inverse $f(a)$ since $f(a).' (f(a))^{-1} = e'$

$\Rightarrow f(a^{-1}) = (f(a))^{-1}$

Ring Isomorphism

Definition : If $(R,+,.)$ and $(R',+',.')$ are two rings, let f be a function from R into R′ i.e.

$f: R \to R'$, then f is called a ring isomorphism if:

1) f is a ring homomorphism. ⇒ 1) $f(a+b)=f(a)+' f(b)$
   2) $f(a.b)=f(a).' f(b)$

2) f is one to one (injective) ⇒ $f(a)=f(b) \Rightarrow a=b \ \forall a,b \in R$

3) f is onto (surjective) ⇒ $\forall\, y \in R' \ \exists\, x \in R \ \ni f(x) = y$

or/ two ring $(R,+,.)$ and $(R', +', .')$ are said to be isomorphic if there exist a one to one ring homomorphism from R onto R' and is denoted by $R \cong R'$.

Example: Let (R,+,.) be a ring with identity and $f_a: R \to R$ defined as

$f_a(x) = axa^{-1} \quad \forall x \in R , a \in R$. show that $R \cong R$.

Sol.: 1) f_a is a ring homo. [راجع محاضرة 15 مثال 3]

2) f is $1-1$ since

$\forall x, y \in R \Rightarrow f_a(x) = f_a(y)$

$\Rightarrow axa^{-1} = aya^{-1}$

$\Rightarrow x = y$

3) f is onto

$\forall\, y \in R \quad \exists\, x = a^{-1}\, y \;\; a \in R$

توضيح

$f_a(x) = y = axa^{-1}$

بالضرب بالطرفين بـ

a, a^{-1}

$x = a^{-1}\, y\, a$

توضيح (2)

$f_a(x) = a(a^{-1}\, y\, a)a^{-1}$

$= y$

H.w./ Let R and R' are rings and Δ_1, Δ_2 are two binary operations of R s.t.

$$x \,\Delta_1\, y = x + y + 1$$

$$x \,\Delta_2\, y = x + y + xy \quad \forall x, y \in R'$$

Show that f: $(R', \Delta_1, \Delta_2) \to (R, +, .)$ is a ring isomorphism s.t

$$f(x) = x + 1 \;\; \forall x \in R'$$