University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 20

Ring Isomorphism

Definition : If (R,+,.) and $(R', +', .')$ are two rings, let f be a function from R into R′ i.e.

$f: R \to R'$, then f is called a ring isomorphism if:

1) f is a ring homomorphism. ⇒ 1) $f(a+b)=f(a)+' f(b)$
   2) $f(a.b)=f(a).' f(b)$

2) f is one to one (injective) ⇒ $f(a)=f(b) \Rightarrow a=b \ \forall a, b \in R$

3) f is onto (surjective) ⇒ $\forall y \in R' \ \exists x \in R \ni f(x) = y$

Question/ Let R and R are rings and Δ_1, Δ_2 are two binary operations of R s.t.

$$x \, \Delta_1 \, y = x + y + 1$$

$$x \, \Delta_2 \, y = x + y + xy \quad \forall x, y \in R$$

Show that f: $(R, \Delta_1, \Delta_2) \rightarrow (R, +, .)$ is a ring isomorphism s.t

$$\mathrm{f}(x) = \mathrm{x} + 1 \;\; \forall x \in \;\; R$$

Solution: 1) Let $x, y \in \mathbb{R}$ then:

L.S) $f(x \, \Delta_1 \, y) =$ $f(x + y + 1)$

$= (x + y + 1) + 1$

$= (x + 1) + (y + 1)$

$= (x + 1) + (y + 1)$

$= f(x) + f(y)$

= R.S

L.S) $f(x \Delta_2 y) =$ $f(x + y + xy)$

$= (x + y + xy) + 1$

$= (x + 1).(y + 1)$

$= f(x).f(y)$

= R.S

∴ f is homo.

2) Let $x, y \in \mathbb{R}$ s.t:

$$f(x) = f(y)$$

$$\rightarrow \quad x+1 = y+1$$

$$\rightarrow \quad x = y$$

$\therefore$ f is $1 - 1$

3) $\forall\ y \in R\ \exists\ \ y - 1 \in R\ \ \ni f(y - 1) = y$

$\therefore$ f is onto

$\therefore$ f is a ring isomorphism

Theorem 12: natural mapping (nat_I) is a homo. From the ring (R,+,.) into the ring (R/I , ⊕ , ⊗) with ker (natI) =I.

Proof: let $nat_I : R \rightarrow R/I$ defined by :

$nat_I(a) = a + I \quad , a \in R$

1) let $a + I, b + I \in R/I \quad \forall\, a, b \in R$

$L.S.)\ nat_I(a + b) = (a + b) + I$

$= (a + I) \oplus (b + I)$

$= nat_I(a) \oplus nat_I(b)$

$= R.S.$

2) $nat_I(ab) = (ab) + I$

$= (a + I) \otimes (b + I)$

$= nat_I(a) \otimes nat_I(b)$

$\therefore nat_I$ is a homo.

$$\ker(\mathrm{nat}_I) = \{a \in R : \mathrm{nat}_I(a) = 0 + I\}$$

$$= \{a \in R : a + I = 0 + I\}$$

$$= \{a \in R : a + I = I\}$$

$$= \{a \in R : a \in I\}$$

$$= I$$

<u>H.w</u>: (Z_4 , +,.) ring, I={0,2}. Show that nat_I is a homo. from $Z_4 \rightarrow Z_4/I$. and find ker (nat_I).