University of Al-Hamdaniya, College of Education
Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 22

Ring Isomorphism

Definition : If $(R,+,.)$ and $(R',+',.')$ are two rings, let f be a function from R into R' i.e.

$f: R \to R'$, then f is called a ring isomorphism if:

1) f is a ring homomorphism. ⇒ 1) $f(a+b)=f(a)+' f(b)$
   2) $f(a.b)=f(a).' f(b)$

2) f is one to one (injective) ⇒ $f(a)=f(b) \Rightarrow a=b \ \forall a,b \in R$

3) f is onto (surjective) ⇒ $\forall y \in R' \ \exists x \in R \ni f(x) = y$

The fundamental theorems of ring homomorphism

Theorem 1) first fund. Theorem

If $f: R \to R'$ is an a ring homo. Then $R/\ker(f) \cong f(R) = R'$

Proof: Define $g: R/\ker(f) \to f(R)$

s.t. $g(x+\ker(f))=f(x)\ \forall\ x \in R$

T.P. g is well- define?

توضيح

To prove $g: R/\ker(f) \to f(R)$ is isomorphism we should prove :

1) G is well define
2) g is a Ring homo.
3) g is 1-1
4) g is onto

Let $x_1 + \ker(f) = x_2 + \ker(f)$ where $x_1 + \ker(f)$ & $x_2 + \ker(f) \in R/\ker(f)$

$\Rightarrow x_1 - x_2 \in \ker(f)$

$\Rightarrow f(x_1 - x_2) = 0'$ (f is homo.)

$\Rightarrow f(x_1) - f(x_2) = 0$

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow g(x_1 + \ker(f)) = g(x_2 + \ker(f))$

∴ g is well define.

T.P. g is homo. ?

Let $x_1 + \ker(f)$, $x_2 + \ker(f) \in R/\ker(f)$

1) $g[(x_1 + \ker(f)) \oplus (x_2 + \ker(f))] =$

$g[(x_1 + x_2) + \ker(f)] = f(x_1 + x_2)$

$= f(x_1) + f(x_2)$

$= g(x_1 + \ker(f)) \oplus g(x_2 + \ker(f))$

2) Let $x_1 + \ker(f), x_2 + \ker(f) \in R/\ker(f)$

$$g[(x_1 + \ker(f)) \otimes (x_2 + \ker(f))] = g[(x_1 . x_2) + \ker(f)]$$

$$= f(x_1 . x_2)$$

$$= f(x_1). f(x_2)$$

$$= g(x_1 + \ker(f)) \otimes g(x_2 + \ker(f))$$

$\therefore$ g is homo.

T.P. g is (1−1). ?

$$g(x_1 + \ker(f)) = g(x_2 + \ker(f))$$

$$\Rightarrow f(x_1) = f(x_2)$$

$$\Rightarrow f(x_1) - f(x_2) = 0$$

$$\Rightarrow f(x_1 - x_2) = 0'$$

$$\therefore x_1 - x_2 \in \ker(f)$$

$$\Rightarrow x_1 + \ker(f) = x_2 + \ker(f)$$

T.P. g is onto ?

Let $y \in f(R)$

$\therefore\ \exists\, x \in R\ s.t.\ f(x) = y$

$\Rightarrow x + \ker(f) \in R/\ker(f)$

$\Rightarrow g(x + \ker(f)) = f(x) = y$

$\therefore R/\ker(f) \cong f(R)$