University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 3

Definition: A ring is a mathematical system (R,*,○) consisting of a nonempty set R and two binary operations * and ○ defined on R such that:

- (R,*) is a commutative group.
- (R, ○) is a semi group.
- The semi group (○) is distributive over the group operation (*).

Example : A ring is an ordered triple (R,+,.) consisting of a nonempty set R and two operations called addition and multiplication respectively satisfying the following:

I. (R,+) is a commutative group

1) $\forall\, a, b \in R\,;\, a + b \in R$ (+closed).

2) $(a + b) + c = a + (b + c) \quad \forall\, a, b, c \in R$ (+associative).

3) $\exists 0 \in R$ s.t $a + 0 = 0 + a = a \;\; \forall\, a \in R$.

4) For each $a \in R, \exists -a \in R$ s.t .

$$a + (-a) = (-a) + a = 0.$$

5) $a + b = b + a \;\; \forall\, a, b \in R$.

II. (R,.) is a Semi group

1) $\forall$ a,b $\in$R ;a.b $\in$R (. closed).

$2)\ (a.b).c = a.(b.c) \quad \forall a,b,c \in R$ (. associative)

III. (.) is distributive over +

1) a . (b + c) = (a . b) + (a . c) [left dist. Law]

2) (b + c) . a = (b . a) + (c . a) [right dist. Law]

$\forall$ a ,b ,c $\in$R.

Definition: A commutative ring is a ring in which (R,○) is a commutative semi group.

Example: (R,+,.) is a commutative ring.

Definition: A ring with identity is a ring in which (R, ○) is a semi group with identity

i.e $\exists\, e \in R$ s.t

$a \circ e = e \circ a = a \quad \forall a \in R.$

Examples:

1) (Z,+ , .) is a commutative ring with identity.

2) (R, +, .) is a commutative ring with identity.

3) (Z_n,+,.) is a comm. ring with identity.

ملاحظة في Z_n شكل العنصر فيها بالشكل [a] وعملية الجمع والضرب

فيها بالشكل

$[a], [b] \in Z_n \rightarrow [a] +_n [b]= [a+b] \in Z_n$

$\forall [a], [b] \in Z_n \rightarrow [a] ._n [b]= [a.b] \in Z_n$

Definition: If A is an arbitrary set, then the set whose elements are all subsets of A is known as the power set of A denoted by p(A):

$$p(A) = \{B: B \subseteq A\}$$

Example: Let A={a,b} show that (p(A),Δ,∩) is a ring. [s.t. (AΔB)=(A – B) ∪ (B – A)

Sol: $A = \{\{a\},\{b\},\{a,b\},\emptyset\}$

Δ	{a}		{b}	{a,b}	∅
{a}	∅		{a,b}	{b}	{a}
{b}	{a,b}		∅	{a}	{b}
{a,b}	{b}		{a}	∅	{a,b}
∅	{a}		{b}	{a,b}	∅

I. (p(A),Δ) is a comm. group

1) Δ is closed , since $\forall \{a\},\{b\} \in p(A) \to \{a\}\Delta\{b\} = \{a,b\} \in p(A)$

2) Δ is asso.

3) $\emptyset$ is an identity

4) $\emptyset^{-1} = \emptyset$, $\{a\}^{-1} = \{a\}$, $\{b\}^{-1} = \{b\}, \{a,b\}^{-1} = \{a,b\}$

5) Δ is comm.

∴ (p(A),Δ) is a comm. group

II. (p(A), ∩) is a semi group

∩	{a}	{b}	{a,b}	∅
{a}	{a}	∅	{a}	∅
{b}				
{a,b}				
∅				

III . $\cap$ is distributive over Δ.

1) $\{a\} \cap (\{b\} \Delta \{a,b\}) = (\{a\} \cap \{b\}) \Delta (\{a\} \cap \{a,b\})$ [left dist. Law]

2) $(\{b\} \Delta \{a,b\}) \cap \{a\} = (\{b\} \cap \{a\}) \Delta (\{a,b\} \cap \{a\})$ [right dist. Law]

$\forall \{a\}, \{b\}, \{a,b\} \in p(A)$

$\therefore (p(A), \Delta, \cap)$ is a ring.