University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 4

Definition: Let (R,+,.) be a ring, an element a ∈ R is called zero divisor if $a \neq 0$

and there exists $b \neq 0$; b ∈ R such that a.b = 0.

Example (1): $(Z_8, +_8, ._8)$ is a ring, zero divisors are {2,4,6}.

.	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

Example (2): $(Z_3, +_3, ._3)$ has no zero divisors.

.	1	2
1	1	2
2	2	1

Note: $(Z_p, +_p, ._p)$, if p is a prime number then there is no zero divisors

Remark: : Let (R,+,.) be a commutative ring with identity 1, if $a \in R$ and $a \neq 0$, a is an invertible element then a is not zero divisor.

That is a is invertible element implies a is not zero divisor.

Proof: if $a \in R$, $a \neq 0$ and a is an invertible element.

$$\therefore \ \exists\, a^{-1} \in R \text{ s.t. } a^{-1}.a = a.a^{-1} = 1$$

Suppose a is a zero divisor. Then $\exists\, b \in R, b \neq 0 \text{ s.t. } a.b = 0$

$$a^{-1}.(a.b) = a^{-1}.0$$

$$\rightarrow (a^{-1}.a).b = 0$$

$$\rightarrow 1.b = 0 \quad \rightarrow b = 0 \ \ C! \ \ (\text{since } b \neq 0)$$

$\therefore$ a is not zero divisor.

Definition: Cancellation Law

Let R be a ring and $a, b, c \in R, a \neq 0$ and $a.b = a.c$ implies $b = c$.

Theorem(1): let (R,+,.) be a commutative ring without zero divisors iff the cancellation law holds for multiplication.

Proof: we assume that R is without zero divisors, and let $a, b, c \in R$ $\;$ s.t. $\; a \neq 0$

$$\rightarrow \quad a.b = a.c$$

$$\rightarrow \quad a.b - a.c = 0$$

$$\rightarrow a(b - c) = 0$$

$\because a \neq 0$ and R without zero divisors

$$\rightarrow b - c = 0$$

$$\rightarrow b = c$$

$\therefore$ the cancellation law holds.

conversely

Suppose that the cancellation law holds and let $a, b \in R$ s.t.

a.b = 0 and let $a \neq 0$ then

a.b = 0 $\rightarrow a.b = a.0$

$\rightarrow b = 0$

Now if $b \neq 0$ then ab = 0

$\rightarrow a.b = 0.b$

$\rightarrow a = 0$

$\therefore$ (R,+,.) is without zero divisors.