

University of Al-Hamdaniya, College of
Education

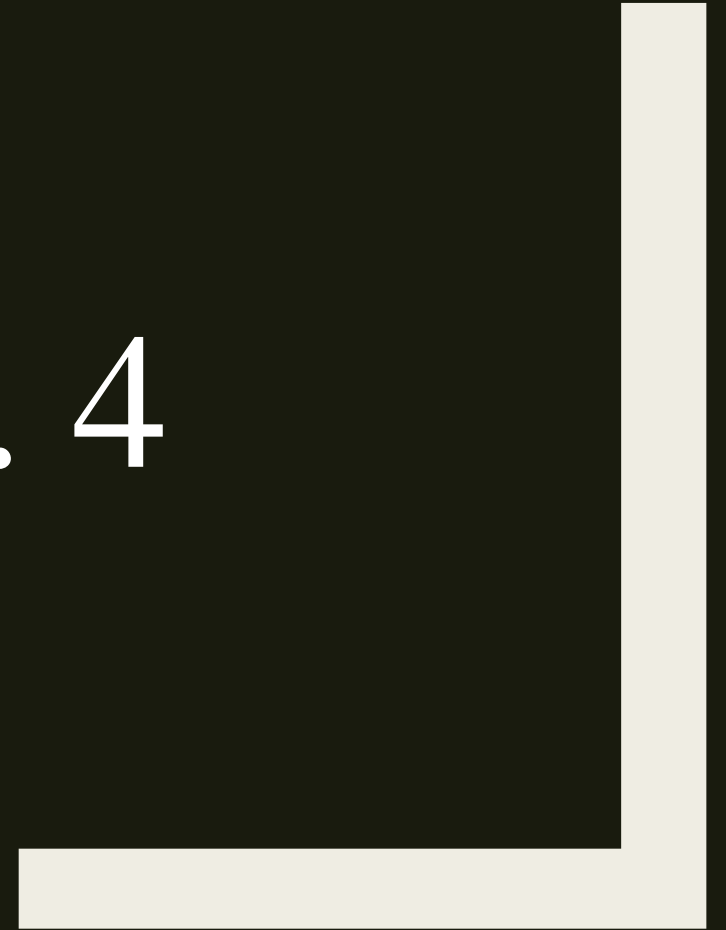
Department of Mathematics

RING THEORY

Level Three

Asst. Lecturer. Hadil Hazim Sami

LECTURE NO. 4



Definition: Let $(R, +, \cdot)$ be a ring, an element $a \in R$ is called zero divisor if $a \neq 0$

and there exists $b \neq 0 ; b \in R$ such that $a \cdot b = 0$.

Example (1): $(\mathbb{Z}_8, +_8, \cdot_8)$ is a ring, zero divisors are $\{2,4,6\}$.

\cdot	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

Example (2): $(\mathbb{Z}_3, +_3, \cdot_3)$ has no zero divisors.

\cdot	1	2
1	1	2
2	2	1

Note: $(\mathbb{Z}_p, +_p, \cdot_p)$, if p is a prime number then there is no zero divisors

Remark: : Let $(R, +, \cdot)$ be a commutative ring with identity 1, if $a \in R$ and $a \neq 0$, a is an invertible element then a is not zero divisor.

That is a is invertible element implies a is not zero divisor.

Proof: if $a \in R$, $a \neq 0$ and a is an invertible element.

$$\therefore \exists a^{-1} \in R \text{ s.t. } a^{-1} \cdot a = a \cdot a^{-1} = 1$$

Suppose a is a zero divisor. Then $\exists b \in R, b \neq 0$ s.t. $a \cdot b = 0$

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$$

$$\rightarrow (a^{-1} \cdot a) \cdot b = 0$$

$$\rightarrow 1 \cdot b = 0 \quad \rightarrow b = 0 \text{ C! (since } b \neq 0)$$

$\therefore a$ is not zero divisor.

Definition: Cancellation Law

Let R be a ring and $a, b, c \in R$, $a \neq 0$ and $a \cdot b = a \cdot c$ implies $b = c$.

Theorem(1): let $(R, +, \cdot)$ be a commutative ring without zero divisors iff the cancellation law holds for multiplication.

Proof: we assume that R is without zero divisors, and let $a, b, c \in R$ s.t. $a \neq 0$

$$\rightarrow a \cdot b = a \cdot c$$

$$\rightarrow a \cdot b - a \cdot c = 0$$

$$\rightarrow a(b - c) = 0$$

$\because a \neq 0$ and R without zero divisors

$$\rightarrow b - c = 0$$

$$\rightarrow b = c$$

\therefore the cancellation law holds.

conversely

Suppose that the cancellation law holds and let $a, b \in R$ s.t.

$a \cdot b = 0$ and let $a \neq 0$ then

$$a \cdot b = 0 \rightarrow a \cdot b = a \cdot 0$$

$$\rightarrow b = 0$$

Now if $b \neq 0$ then $ab = 0$

$$\rightarrow a \cdot b = 0 \cdot b$$

$$\rightarrow a = 0$$

$\therefore (R, +, \cdot)$ is without zero divisors.