



University of Al-Hamdaniya, College of  
Education

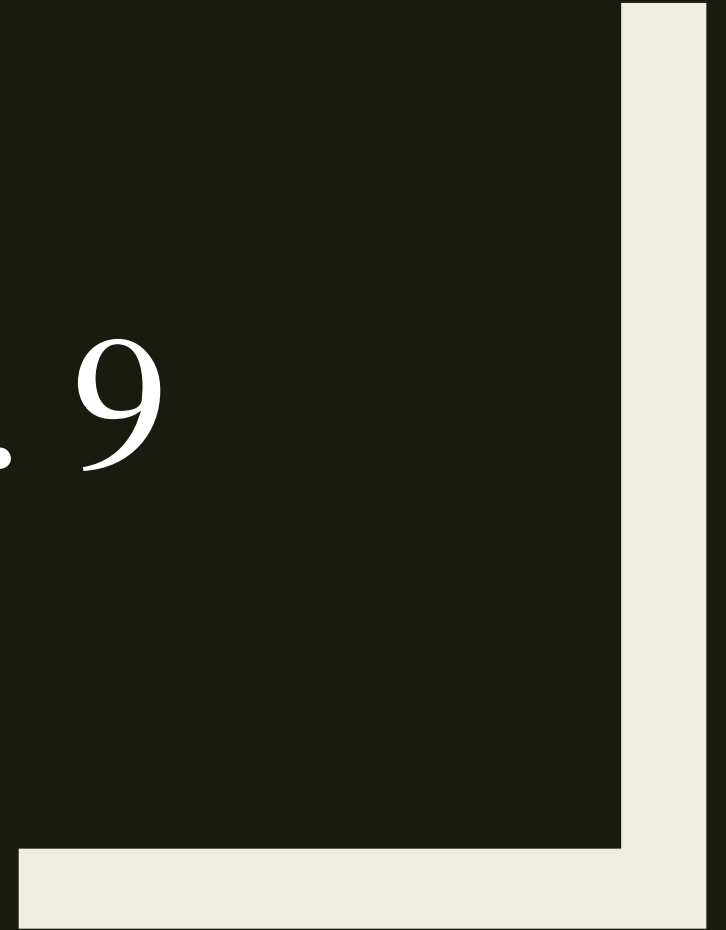
Department of Mathematics

RING THEORY

Level Three

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# LECTURE NO. 9



# Ideals

**Definition:** Let  $(R, +, \cdot)$  be a ring and  $\emptyset \neq I \subseteq R$  then  $(I, +, \cdot)$  is an ideal of  $(R, +, \cdot)$  iff:

1.  $a - b \in I \quad \forall a, b \in I$
2.  $ar \in I$  and  $ra \in I \quad \forall r \in R, a \in I$

**Example:**  $(\mathbb{Z}_{12}, +, \cdot)$  is a ring , ideals of  $(\mathbb{Z}_{12}, +, \cdot)$  are

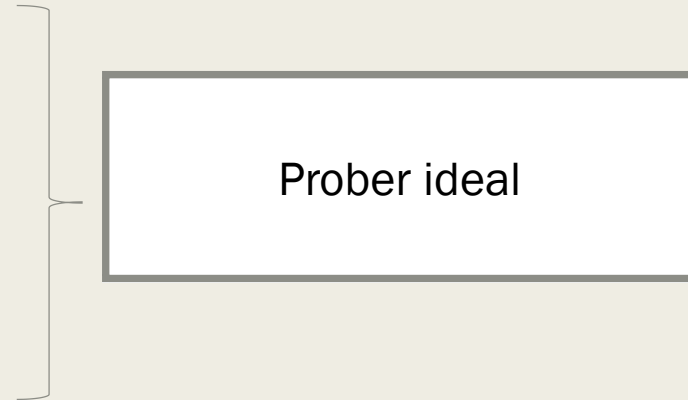
$I_1 = \{0\}$  ,  $I_2 = (\mathbb{Z}_{12}, +, \cdot)$  } trivial ideals

$I_3 = \{0, 2, 4, 6, 8, 10\}$  ,

$I_4 = \{0, 3, 6, 9\}$  ,

$I_5 = \{0, 4, 8\}$  ,

$I_6 = \{0, 6\}$  .



**Definition:** a ring which contains no ideals except the trivial ideals is said to be a simple.

**Example:**  $(\mathbb{Z}_7, +, \cdot)$  is a ring and is a simple.

**Note** every ideal is a subring but converse is not true.

**Example1:**  $(\mathbb{Z}, +, \cdot)$  is a ring ,  $(\mathbb{Z}_e, +, \cdot)$  is an ideal of  $(\mathbb{Z}, +, \cdot)$  and is a subring of  $(\mathbb{Z}, +, \cdot)$ .

**Example2:**  $(\mathbb{Z}, +, \cdot)$  is a subring of  $(\mathbb{R}, +, \cdot)$  but it is not ideal of  $(\mathbb{R}, +, \cdot)$ . Since

$$5 \in \mathbb{Z} \text{ and } \frac{1}{3} \in \mathbb{R} \rightarrow 5 \cdot \frac{1}{3} \notin \mathbb{Z}$$

**Theorem 3:** let  $I$  be a proper ideal of a ring  $(R, +, \cdot)$  with identity then no element of  $I$  has a multiplicative inverse.

**proof:** Let  $0 \neq a \in I$  and suppose that  $a^{-1} \in I$  then  $aa^{-1} = 1 \in I$

Now  $\forall r \in R, r = 1 \cdot r \in I$

$\Rightarrow R \subseteq I, \because I \subseteq R \Rightarrow R=I$  C!

**Example (1):**  $(\mathbb{Z}_8, +_8, \cdot_8)$  is a ring,  $I=\{0,2,4,6\}$  is an ideal of  $\mathbb{Z}_8$

$\cdot$	2	4	6
2	4	0	4
4	0	0	0
6	4	0	4

**Theorem4** : If  $I_1$  and  $I_2$  are two ideals of a ring  $R$ , then  $I_1 \cap I_2$  is also an ideal of  $R$ .

**Proof:** 1)  $I_1 \cap I_2 \neq \emptyset$  (since  $0 \in I_1 \cap I_2$ )

2) let  $a, b \in I_1 \cap I_2 \Rightarrow a, b \in I_1$   $a, b \in I_2$

$\because I_1$  is an ideal of  $R \Rightarrow a - b \in I_1$

$\because I_2$  is an ideal of  $R \Rightarrow a - b \in I_2$

$\therefore a - b \in I_1 \cap I_2$

3)  $\forall r \in R, a \in I_1 \cap I_2 \Rightarrow a \in I_1$  and  $a \in I_2$

$\because I_1$  is an ideal of  $R \Rightarrow a.r, r.a \in I_1$

$\because I_2$  is an ideal of  $R \Rightarrow a.r, r.a \in I_2$

$\because a.r \in I_1 \cap I_2$  &  $r.a \in I_1 \cap I_2$

$\therefore I_1 \cap I_2$  is an ideal of  $R$ .



**Example 1**: let  $(\mathbb{Z}_6, +, \cdot)$  is a ring ,  $I_1 = \{0,2,4\}, I_2 = \{0,3\}$ ,

$I_1$  &  $I_2$  are ideals of  $\mathbb{Z}_6$  then

$I_1 \cap I_2 = \{0\}$  is also ideal of  $\mathbb{Z}_6$ .

$I_1 \cup I_2 = \{0,2,3,4\}$  is not ideals of  $\mathbb{Z}_6$ .

**Example 2:** let  $(\mathbb{Z}_{18}, +, \cdot)$  is a ring then

$$I_1 = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$$

$$I_2 = \{0, 3, 6, 9, 12, 15\}$$

$$I_3 = \{0, 6, 12\}$$

$$I_4 = \{0, 9\}$$

$$I_1 \cap I_2 = I_3$$

$$I_1 \cap I_3 = I_3$$

$$I_2 \cap I_4 = I_4$$

$$I_3 \cap I_4 = \{0\}$$

$\therefore$  If  $(I_i, +, \cdot)$  are ideals of  $(\mathbb{R}, +, \cdot)$

$\therefore (\cap I_i, +, \cdot)$  is also an ideal of  $\mathbb{R}$ .