University of Al-Hamdaniya, College of Education

Department of Mathematics

RING THEORY

Level Three

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LECTURE NO. 9

Ideals

Definition: Let $(R,+,.)$ be a ring and $\emptyset \neq I \subseteq R$ then $(I,+,.)$ is an ideal of $(R,+,.)$ iff:

1. $a-b \in I \ \forall a,b \in I$
2. $ar \in I$ and $ra \in I \ \forall r \in R, a \in I$

Example: $(Z_{12},+,.)$ is a ring , ideals of $(Z_{12},+,.)$ are

$I_1=\{0\}$, $I_2=(Z_{12},+,.)$ } trivial ideals

$I_3=\{0,2,4,6,8,10\}$,

$I_4=\{0,3, 6,9\}$,

$I_5=\{0 ,4, 8\}$,

$I_6=\{0 ,6\}$.

} Prober ideal

<u>Definition:</u> a ring which contains no ideals except the trivial ideals is said to be a simple.

Example: $(Z_7,+,.)$ is a ring and is a simple.

<u>Note</u> every ideal is a subring but converse is not true.

<u>Example1</u>: $(Z,+,.)$ is a ring , $(Z_e,+,.)$ is an ideal of $(Z,+,.)$ and is a subring of $(Z,+,.)$.

<u>Example2:</u> $(Z,+,.)$ is a subring of $(R,+,.)$ but it is not ideal of $(R,+,.)$. Since

$5 \in Z \; and \; \frac{1}{3} \in R \; \rightarrow 5.\frac{1}{3} \notin Z$

Theorem3: let I be a proper ideal of a ring (R,+,.) with identity then no element of I has a multiplicative inverse.

proof: Let $0 \neq a \in I$ and suppose that $a^{-1} \in I$ then $aa^{-1} = 1 \in I$

Now $\forall r \in R, r = 1.r \in I$

$\Rightarrow R \subseteq I, \because I \subseteq R \Rightarrow R=I$ C!

Example (1): $(Z_8 , +_8 , ._8)$ is a ring, I={0,2,4,6} is an ideal of Z_8

.	2	4	6
2	4	0	4
4	0	0	0
6	4	0	4

Theorem4 : If I_1 and I_2 are two ideals of a ring R, then $I_1 \cap I_2$ is also an ideal of R.

Proof: 1) $I_1 \cap I_2 \neq \emptyset$ (since $0 \in I_1 \cap I_2$)

2)let $a, b \in I_1 \cap I_2 \Rightarrow a, b \in I_1$ $a, b \in I_2$

$\because I_1$ is an ideal of R $\Rightarrow a - b \in I_1$

$\because I_2$ is an ideal of R $\Rightarrow a - b \in I_2$

$$\therefore a - b \in I_1 \cap I_2$$

3) $\forall r \in R$, $a \in I_1 \cap I_2 \Rightarrow a \in I_1$ and $a \in I_2$

$\because I_1$ is an ideal of $R \Rightarrow a.r$, $r.a \in I_1$

$\because I_2$ is an ideal of $R \Rightarrow a.r$, $r.a \in I_2$

$\therefore \ a.r \in I_1 \cap I_2$ & $r.a \in I_1 \cap I_2$

$\therefore I_1 \cap I_2$ is an ideal of R.

Example 1: let $(Z_6,+,.)$ is a ring , $I_1 = \{0,2,4\}, I_2 = \{0,3\}$,

I_1 & I_2 are ideals of Z_6 then

$I_1 \cap I_2 = \{0\}$ is also ideal of Z_6.

$I_1 \cup I_2=\{0,2,3,4\}$ is not ideals of Z_6.

Example 2: let $(Z_{18},+,.)$ is a ring then

$$I_1 = \{0,2,4,6,8,10,12,14,16\}$$

$$I_2 = \{0,3,6,9,12,15\}$$

$$I_3 = \{0,6,12\}$$

$$I_4 = \{0,,9\}$$

$$I_1 \cap I_2 = I_3$$

$$I_1 \cap I_3 = I_3$$

$$I_2 \cap I_4 = I_4$$

$$I_3 \cap I_4 = \{0\}$$

$\therefore$ If $(I_i,+,)$ are ideals of (R,+,.)

$$\therefore (\cap I_i ,+,.)\text{is also an ideal of R.}$$