Section Two "Marginal and Conditional Distribution" And Stochastic Independent

Notation

For two r.v's X_1 and X_2 , we shall call

$f(x_1,x_2)$ a joint p.d.f of X_1 and X_2

$F(x_1,x_2)=Pr(X_1 \le x_1 , X_2 \le x_2)$ a joint c.d.f of X_1 and X_2

$M(t_1,t_2)=E(e^{t_1X_1+t_2X_2})$ a joint m.g.f of X_1 and X_2

In general, when we have n r.v's $X_1, X_2, --, X_n$, we shall call

$f(x_1,x_2,--,x_n)$ a joint p.d.f of $X_1, X_2, --, X_n$

$F(x_1,x_2,--,x_n)=Pr(X_1 \le x_1, X_2 \le x_2, ---, X_n \le x_n)$ a joint c.d.f of $X_1, X_2, ---, X_n$

$M(t_1,t_2,---,t_n)=E(e^{t_1X_1+t_2X_2+---+t_nX_n})$ a joint m.g.f of $X_1, X_2, --, X_n$.

Marginal Distributions "In 2-D case"

Defn. Let $f(x_1,x_2)$ be the joint p.d.f of r.v's X_1 and X_2. We define the marginal p.d.f of X_1 and the marginal p.d.f of X_2 as

$$f_1(x_1)=\begin{cases}\sum_{x_2} f(x_1,x_2) & \text{, for disc. Case}\\ \int_{x_2} f(x_1,x_2)\,dx_2 & \text{, for Cont. Case}\end{cases}$$

$$f_2(x_2)=\begin{cases}\sum_{x_1} f(x_1,x_2) & \text{, disc}\\ \int_{x_1} f(x_1,x_2)\,dx_1 & \text{, Cont.}\end{cases}$$

Note

Suppose, we require to evaluate

1. $$Pr(a \le X_1 \le b)=\begin{cases}\sum_{x_1=a}^{b} f_1(x_1)=\sum_{x_1=a}^{b}\sum_{x_2} f(x_1,x_2) & \text{, disc}\\ \int_a^b f_1(x_1)\,dx_1=\int_a^b\int_{x_2} f(x_1,x_2)\,dx_2\,dx_1 & \text{, Cont.}\end{cases}$$

2. $P_r(c \le X_2 \le d) = \begin{cases} \sum_{x_2=c}^{d} f_2(x_2) = \sum_{x_2=c}^{d} \sum_{x_1} f(x_1,x_2) & \\ \int_c^d f_2(x_2)\,dx_2 = \int_c^d \int_{x_1} f(x_1,x_2)\,dx_1\,dx_2 & , \text{con} \end{cases}$

3. $E[u(X_1)] = \begin{cases} \sum_{x_1} u(x_1) f_1(x_1) = \sum_{x_1}\sum_{x_2} u(x_1) f(x_1,x_2) & , \text{disc} \\ \int_{x_1} u(x_1) f_1(x_1)\,dx_1 = \int_{x_1}\int_{x_2} u(x_1) f(x_1,x_2)\,dx_2\,dx_1 & , \text{Cont} \end{cases}$

4. $E[u(X_2)] = \begin{cases} \sum_{x_2} u(x_2) f_2(x_2) = \sum_{x_2}\sum_{x_1} u(x_2) f(x_1,x_2) & , \text{disc} \\ \int_{x_2} u(x_2) f_2(x_2)\,dx_2 = \int_{x_2}\int_{x_1} u(x_2) f(x_1,x_2)\,dx_1\,dx_2 & , \text{cont.} \end{cases}$

Example (1) Let the joint p.d.f of r.v's X and Y be

(i) $f(x,y) = \frac{1}{21}(x+y)$, $x=1,2,3$; $y=1,2$
$= 0$, e.w

(ii) $f(x,y) = e^{-(x+y)}$, $0<x<\infty$, $0<y<\infty$
$= 0$, e.w

(a) Find the marginal p.d.f of X and the marginal p.d.f of Y

(b) Find $E(X)$, $Var(X)$, $E(Y)$, $Var(Y)$, $E(XY)$

Solution (i)

(a) The marginal p.d.f of X is $f_1(x) = \sum_y f(x,y)$

$\Rightarrow f_1(x) = \sum_{y=1}^{2} \frac{1}{21}(x+y) = \frac{1}{21}[(x+1)+(x+2)] = \frac{1}{21}(2x+3)$, $x=1,2,3$
$= 0$, e.w

The marginal p.d.f of Y is $f_2(y) = \sum_x f(x,y)$

$\Rightarrow f_2(y) = \sum_{x=1}^{3} \frac{1}{21}(x+y) = \frac{1}{21}[(1+y)+(2+y)+(3+y)] = \frac{1}{7}(y+2)$, $y=1,2$
$= 0$, e.w

$$\mu_1 = E(X) = \sum_x x f_1(x) = \frac{1}{21}\sum_{x=1}^{3} x(2x+3) = \frac{46}{21}$$

$$E(X^2) = \sum_x x^2 f_1(x) = \frac{1}{21}\sum_{x=1}^{3} x^2(2x+3) = \frac{114}{21}$$

$$\mathrm{Var}(X) = \sigma_1^2 = E(X^2) - \mu_1^2 = \frac{114}{21} - \left(\frac{46}{21}\right)^2 = \frac{278}{441}$$

$$\mu_2 = E(Y) = \sum_y y f_2(y) = \frac{1}{7}\sum_{y=1}^{2} y(y+2) = \frac{11}{7}$$

$$E(Y^2) = \sum_y y^2 f_2(y) = \frac{1}{7}\sum_{y=1}^{2} y^2(y+2) = \frac{19}{7}$$

$$\mathrm{Var}(Y) = \sigma_2^2 = E(Y^2) - \mu_2^2 = \frac{19}{7} - \left(\frac{11}{7}\right)^2 = \frac{12}{49}$$

$$E(XY) = \sum_x \sum_y xy f(x,y) = \frac{1}{21}\sum_{x=1}^{3}\sum_{y=1}^{2} xy(x+y) = \frac{1}{21}\sum_{x=1}^{3}\left[x(x+1) + 2x(x+2)\right]$$

$$= \frac{1}{21}\sum_{x=1}^{3}(3x^2+5x) = \frac{1}{21}(8+22+42) = \frac{24}{7}$$

<u>Solution (ii)</u>

(a) $f_1(x) = \int_y f(x,y)\,dy = \int_0^\infty e^{-(x+y)}\,dy = e^{-x}\int_0^\infty e^{-y}\,dy = -e^{-x}\left(e^{-y}\right|_0^\infty$

$$= -e^{-x}(0-1) = e^{-x} \quad , \quad 0<x<\infty$$
$$= 0 \quad , \text{e.w}$$

Similarly

$$f_2(y) = e^{-y} \quad , \quad 0<y<\infty$$
$$= 0 \quad , \text{e.w}$$

$$\mu_1 = E(X) = \int_x x f_1(x)\,dx = \int_0^\infty x e^{-x}\,dx = 1 = E(Y) = \mu_2$$

$$E(X^2) = \int_x x^2 f_1(x)\,dx = \int_0^\infty x^2 e^{-x}\,dx = 2 = E(Y^2)$$

$$\mathrm{Var}(X) = \sigma_1^2 = E(X^2) - \mu_1^2 = 2 - (1)^2 = 1 = \mathrm{Var}(Y) = \sigma_2^2$$

$$E(XY) = \int_x \int_y xy f(x,y)\,dx\,dy = \int_0^\infty \int_0^\infty xy\, e^{-(x+y)}\,dx\,dy$$

$$= \left(\int_0^\infty x e^{-x} dx\right)\left(\int_0^\infty y e^{-y} dy\right) = \left(\int_0^\infty x e^{-x} dx\right)^2 = (1)^2 = 1$$

The

Defn Let $f(x_1, x_2)$ and $F(x_1, x_2) = Pr(X_1 \le x_1, X_2 \le x_2)$ be the joint p.d.f and the joint c.d.f of r.v's X_1 and X_2. Then the marginal c.d.f of X_1 and the marginal c.d.f of X_2 are respectively

$$F_1(x_1) = F(x_1, \infty) \quad \text{and} \quad F_2(x_2) = F(\infty, x_2)$$

Note If $f_1(x_1)$ and $f_2(x_2)$ are the marginal p.d.f's, then the marginal c.d.f's $F_1(x_1)$ and $F_2(x_2)$ can be obtained as:

$$F_1(x_1) = \begin{cases} \sum_{t_1=-\infty}^{x_1} f_1(t_1), & \text{disc} \\ \int_{-\infty}^{x_1} f_1(t_1)\, dt_1, & \text{Cont.} \end{cases}$$

$$F_2(x_2) = \begin{cases} \sum_{t_2=-\infty}^{x_2} f_2(t_2), & \text{disc} \\ \int_{-\infty}^{x_2} f_2(t_2)\, dt_2, & \text{Cont.} \end{cases}$$

Example (2) Back to Example (1) and find the joint c.d.f of X and Y. Also find the marginal c.d.f's.

(i) We have $f(x,y) = \frac{1}{21}(x+y)$, $x = 1, 2, 3$, $y = 1, 2$
$= 0$, e.w

$$F(x,y) = \sum_{t=-\infty}^{x} \sum_{s=-\infty}^{y} f(t,s) = \sum_{t=1}^{x} \sum_{s=1}^{y} \frac{1}{21}(t+s) = \frac{1}{21}\sum_{t=1}^{x}\left[(t+1)+(t+2)+\cdots+(t+y)\right]$$

$$= \frac{1}{21}\sum_{t=1}^{x}\left[yt + \frac{y(y+1)}{2}\right] = \frac{1}{21}\left[\frac{yx(x+1)}{2} + \frac{xy(y+1)}{2}\right] = \frac{1}{42}xy(x+y+2)$$

$$F(x,y)=\begin{cases}0 & , \quad x<1,\ y<1\\ \frac{1}{42}xy(x+y+2) & , \quad 1\le x<3,\ 1\le y<2\\ 1 & , \quad x\geq 3,\ y\geq 2\end{cases}$$

The marginal c.d.f of X and the marginal c.d.f of Y are :-

$$F_1(x)=F(x,2)=\begin{cases}0 & , \quad x<1\\ \frac{1}{21}x(x+4) & , \quad 1\le x<3\\ 1 & , \quad x\geq 3\end{cases}$$

$$F_2(x)=F(3,y)=\begin{cases}0 & , \quad y<1\\ \frac{1}{14}y(y+5) & , \quad 1\le y<2\\ 1 & , \quad y\geq 2\end{cases}$$

(ii) $f(x,y)=e^{-(x+y)}$, $0<x<\infty$, $0<y<\infty$

$= 0$, e.w

$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f(t,s)\,dt\,ds=\int_{0}^{x}\int_{0}^{y} e^{-(t+s)}\,ds\,dt=\underbrace{\left(\int_0^{\infty} e^{-x}dx\right)\left(\int_0^{y} e^{-s}dy\right)}$$

$$=\left(-e^{-t}\Big|_0^{x}\right)\left(-e^{-s}\Big|_0^{y}\right)=\begin{cases}0 & , \quad x\le 0,\ y\le 0\\ (1-e^{-x})(1-e^{-y}) & , \quad 0<x<\infty,\ 0<y<\infty\\ 1 & , \quad x=y=\infty\end{cases}$$

$$F_1(x)=F(x,\infty)=\begin{cases}0 & , \quad x\le 0\\ 1-e^{-x} & , \quad 0<x<\infty\\ 1 & , \quad x=\infty\end{cases}$$

$$F_2(y)=F(\infty,y)\quad\begin{cases}0 & , \quad y\le 0\\ 1-e^{-y} & , \quad 0<x<\infty\\ 1 & , \quad y=\infty\end{cases}$$

The Covariance and The Correlation Coefficient

Let the r.v's X and Y have means μ_1,μ_2 and variances σ_1^2,σ_2^2 respectively. The mathematical expectation $E[(X-\mu_1)(Y-\mu_2)]$ is called the covariance of X and Y, and denoted by $cov(X,Y)$.

Note:
$$\begin{aligned}
cov(X,Y) &= E[(X-\mu_1)(Y-\mu_2)] = E(XY - \mu_1 Y - \mu_2 X + \mu_1\mu_2) \\
&= E(XY) - \mu_1 E(Y) - \mu_2 E(X) + \mu_1\mu_2 \\
&= E(XY) - \mu_1\mu_2 - \mu_2\mu_1 + \mu_1\mu_2 = E(XY) - \mu_1\mu_2
\end{aligned}$$

$\rho = \frac{Cov(X,Y)}{\sigma_1 \sigma_2}$

Defn The correlation coefficient between two r.v's X and Y, denoted by $\rho = \rho(X,Y)$, is defined to be

$$\rho = \frac{cov(X,Y)}{\sigma_1 \sigma_2} = \frac{E(XY) - \mu_1\mu_2}{\sigma_1\sigma_2}$$

Note: $-1 \le \rho \le 1$

Example (3) Let the joint p.d.f of r.v's X and Y be

$$f(x,y) = x+y \quad , \ 0<x<1 \ , \ 0<y<1$$
$$= 0 \quad , \ e.w$$

Compute the correlation coefficient between X and Y.

Solution

the marginal p.d.f of X is

$$f_1(x) = \int_y f(x,y)\,dy = \int_{y=0}^{1} (x+y)\,dy = xy + \tfrac{1}{2}y^2 \Big|_0^1 = x + \tfrac{1}{2} \quad , \ 0<x<1$$
$$= 0 \quad , \ e.w$$

Similarly, the marginal p.d.f of Y is

$$f_2(y) = y + \tfrac{1}{2} \quad , \ 0<y<1$$
$$= 0 \quad , \ e.w$$

$$\mu_1 = E(X) = \int_x x f_1(x)\,dx = \int_0^1 x(x+\tfrac{1}{2})\,dx = \frac{x^3}{3} + \frac{x^2}{4}\Big|_0^1 = \frac{7}{12} = E(Y) = \mu_2$$

$$E(X^2) = \int_x x^2 f_1(x)\,dx = \int_0^1 x^2(x+\tfrac{1}{2})\,dx = \frac{x^4}{4} + \frac{x^3}{6}\Big|_0^1 = \frac{5}{12}$$

$$\sigma_1^2 = Var(X) = E(X^2) - \mu_1^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144} = Var(Y) = \sigma_2^2$$

$$E(XY) = \int_x\int_y xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy\,(x+y)\,dy\,dx = \int_0^1\int_0^1 (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\Big|_0^1\right) dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right) dx = \frac{x^3}{6} + \frac{x^2}{6}\Big|_0^1 = \frac{1}{3}$$

$Cov(X,Y) = E(XY) - \mu_1\mu_2 = \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144}$

$$\rho = \frac{Cov(X,Y)}{\sigma_1 \sigma_2} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = -\frac{1}{11}$$

Joint Moment Generating Function " In two - D case "

Let $f(x,y)$ be the joint p.d.f of r.v's X, Y. The mathematical expectation $E(e^{t_1X+t_2Y})$, if it is exist for $-h_1<t_1<h_1$, $-h_2<t_2<h_2$, is called the joint m.g.f of X and Y, and denoted by $M(t_1,t_2)$.

That is

$$M(t_1,t_2) = E(e^{t_1X+t_2Y}) = \begin{cases} \sum_x \sum_y e^{t_1x+t_2y} f(x,y) & \text{, for disc. case} \\ \int_x \int_y e^{t_1x+t_2y} f(x,y)\, dx\, dy & \text{, for cont. case} \end{cases}$$

If we set $t_1 = t_2 = 0$, we have $M(0,0) = 1$

Notes

(i) The marginal m.g.f of X and the marginal m.g.f of Y can be obtained as

$M(t_1,0) = E(e^{t_1X}) = M_1(t_1)$

$M(0,t_2) = E(e^{t_2Y}) = M_2(t_2)$

(ii) If we differentiate $M(t_1,t_2)$ partially k times w.r.to t_1 and m times w.r.to t_2, we have

$$\frac{\partial^{k+m} M(t_1,t_2)}{\partial t_2^m\, \partial t_1^k} = \begin{cases} \sum_x \sum_y x^k y^m e^{t_1x+t_2y} f(x,y) & \text{, for disc case.} \\ \int_x \int_y x^k y^m e^{t_1x+t_2y} f(x,y)\, dx\, dy & \text{, for cont. case.} \end{cases}$$

set $t_1 = t_2 = 0$, we have

$$\left.\frac{\partial^{k+m} M(t_1,t_2)}{\partial t_2^m\, \partial t_1^k}\right|_{t_1=t_2=0} = \begin{cases} \sum_x \sum_y x^k y^m f(x,y) \\ \int_x \int_y x^k y^m f(x,y)\, dx\, dy \end{cases} = E(X^k Y^m)$$

In particular

$$\frac{\partial M(t_1,t_2)}{\partial t_1}\bigg|_{t_1=t_2=0} = \frac{\partial M(0,0)}{\partial t_1} = E(X) = \mu_1$$

$$\frac{\partial M(t_1,t_2)}{\partial t_2}\bigg|_{t_1=t_2=0} = \frac{\partial M(0,0)}{\partial t_2} = E(Y) = \mu_2$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_1^2}\bigg|_{t_1=t_2=0} = \frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) \Rightarrow var(X) = \sigma_1^2 = E(X^2) - \mu_1^2$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_2^2}\bigg|_{t_1=t_2=0} = \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(Y^2) \Rightarrow var(Y) = \sigma_2^2 = E(Y^2) - \mu_2^2$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_2 \partial t_1}\bigg|_{t_1=t_2=0} = \frac{\partial^2 M(0,0)}{\partial t_2 \partial t_1} = E(XY)$$

Then

$$cov(X,Y) = E(XY) - \mu_1\mu_2 \quad \text{and} \quad \rho = \rho(X,Y) = \frac{cov(X,Y)}{\sigma_1 \sigma_2}$$

(iii) If we set $\psi(t_1,t_2) = \ln M(t_1,t_2)$, then

$$\frac{\partial \psi(t_1,t_2)}{\partial t_i} = \frac{\frac{\partial M(t_1,t_2)}{\partial t_i}}{M(t_1,t_2)} \quad , \quad i=1,2$$

$$\Rightarrow \frac{\partial \psi(0,0)}{\partial t_i} = \frac{\frac{\partial M(0,0)}{\partial t_i}}{M(0,0)} = \frac{\mu_i}{1} = \mu_i \quad , i=1,2$$

$$\frac{\partial^2 \psi(t_1,t_2)}{\partial t_i^2} = \frac{M(t_1,t_2)\frac{\partial^2 M(t_1,t_2)}{\partial t_i^2} - \frac{\partial M(t_1,t_2)}{\partial t_i}\cdot\frac{\partial M(t_1,t_2)}{\partial t_i}}{[M(t_1,t_2)]^2} \quad , i=1,2$$

$$\frac{\partial^2 \psi(0,0)}{\partial t_i^2} = \frac{M(0,0)\frac{\partial^2 M(0,0)}{\partial t_i^2} - \frac{\partial M(0,0)}{\partial t_i}\cdot\frac{\partial M(0,0)}{\partial t_i}}{[M(0,0)]^2}$$

$$= \frac{(1)\,E(X_i^2) - E(X_i)\cdot E(X_i)}{(1)^2} = \sigma_i^2 \ , \ i=1,2$$

$$\frac{\partial^2 \psi(t_1,t_2)}{\partial t_2 \partial t_1} = \frac{M(t_1,t_2)\dfrac{\partial^2 M(t_1,t_2)}{\partial t_2 \partial t_1} - \dfrac{\partial M(t_1,t_2)}{\partial t_1}\cdot\dfrac{\partial M(t_1,t_2)}{\partial t_2}}{[M(t_1,t_2)]^2}$$

$$\frac{\partial^2 \psi(0,0)}{\partial t_2 \partial t_1} = \frac{(1)\,E(XY) - E(X)\cdot E(Y)}{(1)^2} = Cov(X,Y)$$

Example (4) Let the joint p.d.f of r.v X and Y be

$$f(x,y) = e^{-y} \quad , 0<x<y<\infty$$
$$= 0 \quad , e.w$$

(1) Find the marginal p.d.f of X and the marginal p.d.f of Y.

(2) Find $\mu_1 = E(X)$, $\sigma_1^2 = var(X)$, $\mu_2 = E(Y)$, $\sigma_2^2 = var(Y)$, $E(XY)$, $cov(X,Y)$, and $\rho(X,Y)$ by using direct expectation approach.

(3) Find the joint c.d.f of X and Y

(4) Find the marginal c.d.f of X and the marginal c.d.f of Y

(5) Find the joint m.g.f of X and Y

(6) Find the marginal m.g.f of X and the marginal m.g.f of Y

(7) Use the joint m.g.f and find $E(X)$, $Var(X)$, $E(Y)$, $Var(Y)$, $Cov(X,Y)$ and $\rho(X,Y)$

Solution

(1) The marginal p.d.f of X is $f_1(x) = \int_y f(x,y)\,dy = \int_{y=x}^{\infty} e^{-y}\,dy$

$$f_1(x) = -e^{-y}\Big|_x^{\infty} = -(e^{-\infty} - e^{-x}) = e^{-x} \quad , 0<x<\infty$$
$$= 0 \quad , e.w$$

The marginal p.d.f of Y is $f_2(y) = \int_x f(x,y)\,dx = \int_{x=0}^{y} e^{-y}\,dx = e^{-y}x\Big|_0^y$

$$f_2(y) = e^{-y}(y-0) = y e^{-y} \quad , 0<y<\infty$$
$$= 0 \quad , e.w$$

(2) $\mu_1 = E(X) = \int_x x f_1(x)\,dx = \int_0^{\infty} x e^{-x}\,dx = 1$

$$E(X^2) = \int_x x^2 f_1(x)\,dx = \int_0^{\infty} x^2 e^{-x}\,dx = 2$$

$$\Rightarrow \sigma_1^2 = Var(X) = E(X^2) - \mu_1^2 = 2 - (1)^2 = 1$$

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$M(t_1,t_2)$

$$\mu_2 = E(Y) = \int_y y f_2(y)\,dy = \int_0^\infty y\cdot y e^{-y}\,dy = \int_0^\infty y^2 e^{-y}\,dy = 2$$

$$E(Y^2) = \int_y y^2 f_2(y)\,dy = \int_0^\infty y^2\cdot y e^{-y}\,dy = \int_0^\infty y^3 e^{-y}\,dy = 6$$

$$\Rightarrow \sigma_2^2 = Var(Y) = E(Y^2) - \mu_2^2 = 6-(2)^2 = 2$$

$$E(XY) = \int_x\int_y xy\, f(x,y)\,dx\,dy = \int_{x=0}^{\infty}\int_{y=x}^{\infty} xy\, e^{-y}\,dy\,dx \overset{or}{=} \int_{y=0}^{\infty}\int_{x=0}^{y} xy\, e^{-y}\,dx\,dy$$

$$= \int_{y=0}^{\infty} y e^{-x}\left(\frac{x^2}{2}\Big|_0^y\right) dy = \frac{1}{2}\int y^3 e^{-y}\,dy = \frac{6}{2} = 3$$

$$Cov(X,Y) = E(XY) - \mu_1\mu_2 = 3-(1)(2) = 1$$

$$\rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_1\sigma_2} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(3) The joint c.d.f of X and Y is

$$F(x,y) = Pr(X\le x, Y\le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(t,s)\,dt\,ds = \int_{t=0}^{x}\int_{s=t}^{y} e^{-s}\,ds\,dt$$

$$= \int_{t=0}^{x}\left(-e^{-s}\Big|_t^y\right) dt = \int_{t=0}^{x}(e^{-t}-e^{-y})\,dt = -e^{-t} - e^{-y}t\Big|_0^x$$

$$= (-e^{-x} - xe^{-y}) - (-1-0) = \begin{cases} 0, & x\le 0,\ y\le 0 \\ 1-e^{-x}-xe^{-y}, & 0<x<y<\infty \\ 1, & x=y,\ y=\infty \end{cases}$$

(4) The marginal c.d.f of X and the marginal c.d.f of Y are:-

$$F_1(x) = F(x,\infty) = \begin{cases} 0, & x\le 0 \\ 1-e^{-x}, & 0<x<\infty \\ 1, & x=\infty \end{cases},\quad F_2(y) = F(y,y) = \begin{cases} 0, & y\le 0 \\ 1-e^{-y}-ye^{-y}, & 0<y<\infty \\ 1, & y=\infty \end{cases}$$

(5) The joint m.g.f of X and Y is

$$M(t_1,t_2) = E\left(e^{t_1X+t_2Y}\right) = \int_x\int_y e^{t_1x+t_2y} f(x,y)\,dx\,dy = \int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{t_1x+t_2y}\, e^{-y}\,dy\,dx$$

$$= \int_{x=0}^{\infty} e^{t_1x}\left\{\int_{y=x}^{\infty} e^{-(1-t_2)y}\,dy\right\}dx = \int_{x=0}^{\infty} e^{t_1x}\left(-\frac{1}{1-t_2} e^{-(1-t_2)y}\Big|_x^{\infty}\right)dx$$

$$= -\frac{1}{1-t_2}\int_{x=0}^{\infty} e^{t_1x}\left[0 - e^{-(1-t_2)x}\right]dx = \frac{1}{1-t_2}\int_{x=0}^{\infty} e^{-(1-t_1-t_2)x}\,dx$$

$$= -\frac{1}{(1-t_2)(1-t_1-t_2)} e^{-(1-t_1-t_2)x}\Big|_0^{\infty} = -\frac{1}{(1-t_2)(1-t_1-t_2)}(0-1)$$

$$\therefore M(t_1,t_2) = \frac{1}{(1-t_2)(1-t_1-t_2)} = (1-t_2)^{-1}(1-t_1-t_2)^{-1}, \quad t_2<1 \text{ and } t_1+t_2<1$$

check : $M(0,0) = 1$

(6) The marginal m.g.f of X and the marginal m.g.f of Y are :-

$$M_1(t_1) = M(t_1,0) = \frac{1}{(1-t_1)} = (1-t_1)^{-1}, \quad M_2(t_2) = M(0,t_2) = \frac{1}{(1-t_2)^2} = (1-t_2)^{-2}$$

(7) $$\frac{\partial M(t_1,t_2)}{\partial t_1} = (1-t_2)^{-1}(1-t_1-t_2)^{-2} \Rightarrow \frac{\partial M(0,0)}{\partial t_1} = 1 = E(X) = \mu_1$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_1^2} = 2(1-t_2)^{-1}(1-t_1-t_2)^{-3} \Rightarrow \frac{\partial^2 M(0,0)}{\partial t_1^2} = 2 = E(X^2)$$

$$\Rightarrow \sigma_1^2 = Var(X) = E(X^2) - \mu_1^2 = 2-(1)^2 = 1$$

$$\frac{\partial M(t_1,t_2)}{\partial t_2} = (1-t_2)^{-1}(1-t_1-t_2)^{-2} + (1-t_1-t_2)^{-1}(1-t_2)^{-2} \Rightarrow \frac{\partial M(0,0)}{\partial t_2} = 1+1 = 2$$
$$= E(Y) = \mu$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_2^2} = 2(1-t_2)^{-1}(1-t_1-t_2)^{-3} + (1-t_1-t_2)^{-2}(1-t_2)^{-2} + 2(1-t_1-t_2)^{-1}(1-t_2)^{-3} +$$
$$+ (1-t_2)^{-2}(1-t_1-t_2)^{-2}$$

$$\Rightarrow \frac{\partial^2 M(0,0)}{\partial t_2^2} = 2+1+2+1 = 6 = E(Y^2)$$

$$\Rightarrow \sigma_2^2 = Var(Y) = E(Y^2) - \mu_2^2 = 6-(2)^2 = 2$$

$$\frac{\partial^2 M(t_1,t_2)}{\partial t_2 \partial t_1} = \frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2} = 2(1-t_2)^{-1}(1-t_1-t_2)^{-3} + (1-t_1-t_2)^{-2}(1-t_2)^{-2}$$

$$\Rightarrow \frac{\partial^2 \psi(0,0)}{\partial t_2 \partial t_1} = 2+1 = 3 = E(XY)$$

$$\Rightarrow Cov(X,Y) = E(XY) - \mu_1\mu_2 = 3-(1)(2) = 1$$

$$\Rightarrow \rho = \rho(X,Y) = \frac{cov(X,Y)}{\sigma_1\sigma_2} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

OR

$$\psi(t_1,t_2) = \ln M(t_1,t_2) = -\ln(1-t_2) - \ln(1-t_1-t_2)$$

$$\frac{\partial \psi(t_1,t_2)}{\partial t_1} = -\frac{(-1)}{1-t_1-t_2} = (1-t_1-t_2)^{-1} \Rightarrow \frac{\partial \psi(0,0)}{\partial t_1} = 1 = E(X) = \mu_1$$

$$\frac{\partial^2 \psi(t_1,t_2)}{\partial t_1^2} = (1-t_1-t_2)^{-2} \Rightarrow \frac{\partial^2 \psi(0,0)}{\partial t_1^2} = 1 = Var(X) = \sigma_1^2$$

$$\frac{\partial \psi(t_1,t_2)}{\partial t_2} = -\frac{(-1)}{(1-t_2)} - \frac{(-1)}{1-t_1-t_2} = (1-t_2)^{-1} + (1-t_1-t_2)^{-1}$$

$$\Rightarrow \frac{\partial \psi(0,0)}{\partial t_2} = 1+1 = 2 = E(Y) = \mu_2$$

$$\frac{\partial^2 \psi(t_1,t_2)}{\partial t_2^2} = (1-t_2)^{-2} + (1-t_1-t_2)^{-2} \Rightarrow \frac{\partial^2 \psi(0,0)}{\partial t_2^2} = 1+1 = 2 = Var(Y) = \sigma_2^2$$

$$\frac{\partial \psi(t_1,t_2)}{\partial t_2 \partial t_1} = \frac{\partial^2 \psi(t_1,t_2)}{\partial t_1 \partial t_2} = (1-t_1-t_2)^{-2} \Rightarrow \frac{\partial^2 \psi(0,0)}{\partial t_2 \partial t_1} = 1 = cov(X,Y)$$

$$\Rightarrow \rho = \rho(X,Y) = \frac{cov(X,Y)}{\sigma_1\sigma_2} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

"In 2-D space"

Conditional Distribution

Defn

Let $f_1(x)$, $f_2(y)$, and $f(x,y)$ be respectively the marginal p.d.f of X, the marginal p.d.f of Y, and the joint p.d.f of X and Y. We define the conditional p.d.f of X given $Y=y$ as

$$f_1(x|y) = \frac{f(x,y)}{f_2(y)} \quad , \quad f_2(y) \neq 0$$

and the conditional p.d.f of Y given $X=x$ as

$$f_2(y|x) = \frac{f(x,y)}{f_1(x)} \quad , \quad f_1(x) \neq 0$$

To show that $f_1(x|y)$ is valid p.d.f, we note that

(i) $f_1(x|y) \geq 0$, because $f(x,y) \geq 0$ and $f_2(y) > 0$

(ii) $\sum_x f_1(x|y) = \sum_x \frac{f(x,y)}{f_2(y)} = \frac{1}{f_2(y)} \sum_x f(x,y) = \frac{f_2(y)}{f_2(y)} = 1$

or

$\int_x f_1(x|y)\,dx = \int_x \frac{f(x,y)}{f_2(y)}\,dx = \frac{1}{f_2(y)} \int_x f(x,y)\,dx = \frac{f_2(y)}{f_2(y)} = 1$

Notes

(1)
$$\Pr(a \leq X \leq b \mid Y=y) = \begin{cases} \sum_{x=a}^{b} f_1(x|y) & , \text{ for disc. case} \\ \int_{x=a}^{b} f_1(x|y)\,dx & , \text{ for cont. case} \end{cases}$$

$$\Pr(c \leq Y \leq d \mid X=x) = \begin{cases} \sum_{y=c}^{d} f_2(y|x) & , \text{ disc} \\ \int_{y=c}^{d} f_2(y|x)\,dy & , \text{ cont.} \end{cases}$$

(2) If $u(X)$ is a function of X alone, then the conditional expectat of $u(X)$ given $Y=y$ is

$$E[u(X) \mid Y=y] = \begin{cases} \sum_x u(x) f_1(x|y) & , \text{ disc} \\ \int_x u(x) f_1(x|y)\,dx & , \text{ cont.} \end{cases}$$

If $u(Y)$ is a function of Y alone, then the conditional of $u(Y)$ given $X=x$ is

$$E[u(Y)|X=x] = \begin{cases} \sum_y u(y)\, f_2(y|x) & , \text{ disc} \\ \int_x u(y)\, f_2(y|x)\, dy & , \text{ cont.} \end{cases}$$

In particular the conditional mean and the conditional variance of X given $Y=y$

$$E[X|Y=y] = \mu_{x|y} = \begin{cases} \sum_x x\, f_1(x|y) & , \text{ disc} \\ \int_x x\, f_1(x|y)\, dx & , \text{ Cont.} \end{cases}$$

$$E[X^2|Y=y] = \begin{cases} \sum_x x^2\, f_1(x|y) & , \text{ disc} \\ \int_x x^2\, f_1(x|y)\, dx & , \text{ Cont.} \end{cases}$$

and

$$Var(X|Y=y) = \sigma^2_{x|y} = E[X^2|Y=y] - \mu^2_{x|y}$$

Similarly the conditional mean and the conditional variance of Y given $X=x$

$$E[Y|X=x] = \mu_{y|x} = \begin{cases} \sum_y y\, f_2(y|x) & , \text{ disc} \\ \int_y y\, f_2(y|x)\, dy & , \text{ Cont.} \end{cases}$$

$$E[Y^2|X=x] = \begin{cases} \sum_y y^2\, f_2(y|x) & , \text{ disc} \\ \int_y y^2\, f_2(y|x)\, dy & , \text{ cont.} \end{cases}$$

and

$$Var(Y|X=x) = \sigma^2_{y|x} = E[Y^2|X=x] - \mu^2_{y|x}$$

Example (5) Back to Example (4) and find
(1) The conditional p-d-f of X given $Y=y$
(2) The conditional p-d-f of Y given $X=x$
(3) $E(X|Y=y)$ and $Var(X|Y=y)$
(4) $E(Y|X=x)$ and $Var(Y|X=x)$, (5) $Pr(1<X<2)$ and $Pr(1<X<2|Y=2)$

Solution

we have $f(x,y) = e^{-y}$, $0<x<y<\infty$
$= 0$, e.w

$f_1(x) = e^{-x}$, $0<x<\infty$, $f_2(y) = y e^{-y}$, $0<y<\infty$
$= 0$, e.w $= 0$, e.w

(1) The conditional p-d-f of X given $Y=y$ is

$$f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{e^{-y}}{y e^{-y}} = \frac{1}{y}, \quad 0<x<y \text{ for any } 0<y<\infty$$
$$= 0, \quad \text{e.w}$$

(2) The conditional p-d-f of Y Given $X=x$ is

$$f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)}, \quad x<y<\infty \text{ for any } 0<x<\infty$$
$$= 0, \quad \text{e.w}$$

(3) $$\mu_{x|y} = E(X|Y=y) = \int_x x f_1(x|y)\,dx = \int_{x=0}^{y} x \frac{1}{y}\,dx = \frac{1}{y}\left(\frac{x^2}{2}\Big|_0^y\right) = \frac{y}{2}$$

$$E(X^2|Y=y) = \int_x x^2 f_1(x|y)\,dx = \int_{x=0}^{y} x^2 \frac{1}{y}\,dx = \frac{1}{y}\left(\frac{x^3}{3}\Big|_0^y\right) = \frac{y^2}{3}$$

$$\sigma^2_{x|y} = Var(X|Y=y) = E(X^2|Y=y) - \mu^2_{x|y} = \frac{y^2}{3} - \left(\frac{y}{2}\right)^2 = \frac{y^2}{12}$$

(4) $$\mu_{y|x} = E(Y|X=x) = \int_y y f_2(y|x)\,dy = \int_{y=x}^{\infty} y e^{-(y-x)}\,dx = e^{x}\int_x^{\infty} y e^{-y}\,dy$$
$$= e^{x}\left(-y e^{-y} - e^{-y}\Big|_x^{\infty}\right) = e^{x}(x e^{-x} + e^{-x}) = x+1$$

$$E(Y^2|X=x) = \int_y y^2 f_2(y|x)\,dy = \int_{y=x}^{\infty} y^2 e^{-(y-x)}\,dy = e^{x}\int_x^{\infty} y^2 e^{-y}\,dy$$
$$= e^{x}\left(-y^2 e^{-y} - 2y e^{-y} - 2e^{-y}\Big|_x^{\infty}\right) = e^{x}(x^2 e^{-x} + 2x e^{-x} + 2e^{-x}) = x^2+2x+2$$

$$\sigma^2_{y|x} = Var(Y|X=x) = E(Y^2|X=x) - \mu^2_{y|x} = x^2+2x+2-(x+1)^2$$
$$= x^2+2x+2-x^2-2x-1 = 1$$

(5) $$Pr(1<X<2) = \int_{x=1}^{2} f_1(x)\,dx = \int_1^2 e^{-x}\,dx = -e^{-x}\Big|_1^2 = e^{-1}-e^{-2}$$

$$Pr(1<X<2|Y=2) = \int_{x=1}^{2} f_1(x|y=2)\,dx = \int_1^2 \frac{1}{2}\,dx = \frac{x}{2}\Big|_1^2 = \frac{1}{2}(2-1) = \frac{1}{2}$$

<u>Example (6)</u> Let the r.v X has p.d.f $f(x)$ and c.d.f $F(x)$.
Define
$$f(x|X>x_0) = \frac{f(x)}{1-F(x_0)}, \quad x_0<x<\infty, \quad x_0 \text{ fixed number}$$
$$= 0, \quad \text{e.w}, \quad F(x_0) \neq 1$$

(a) Show that $f(x|X>x_0)$ be a conditional p.d.f of X given $X>x_0$

(b) Let $f(x) = e^{-x}$, $0<x<\infty$
$= 0$, e.w. Compute $Pr(X>2|X>1)$

<u>Solution</u>

(a) (i) $f(x|X>x_0) \geq 0$, yes, because $f(x) \geq 0$ and $1-F(x_0)>0$

(ii) $$\int_{x_0}^{\infty} f(x|X>x_0)\,dx = \int_{x_0}^{\infty} \frac{f(x)}{1-F(x_0)}\,dx = \frac{1}{1-F(x_0)} \int_{x_0}^{\infty} f(x)\,dx$$
$$= \frac{1}{1-F(x_0)}\, F(x)\Big|_{x_0}^{\infty} = \frac{F(\infty)-F(x_0)}{1-F(x_0)} = \frac{1-F(x_0)}{1-F(x_0)} = 1$$

(b) $f(x) = e^{-x}$, $0<x<\infty$ $\Rightarrow$ $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x e^{-t}\,dt = -e^{-t}\Big|_0^x$$
$= 0$, e.w $$= 1-e^{-x}$$

$$f(x|X>1) = \frac{f(x)}{1-F(1)} = \frac{e^{-x}}{1-1+e^{-1}} = e \cdot e^{-x}, \quad 1<x<\infty$$
$$= 0, \quad \text{e.w}$$

$$Pr(X>2|X>1) = \int_2^{\infty} f(x|X>1)\,dx = \int_2^{\infty} e \cdot e^{-x}\,dx = -e\left(e^{-x}\Big|_2^{\infty}\right)$$
$$= -e(0-e^{-2}) = e^{-1}$$

Stochastic Independent

Defn Let X_1 and X_2 be two r.v's having joint p.d.f $f(x_1, x_2)$ and marginal p.d.f's $f_1(x_1)$ and $f_2(x_2)$ respectively. The r.v's X_1 and X_2 are said to be sto. indep. if and only if

$$f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

. otherwise X_1 and X_2 are said to be sto. dep.

Note

If the r.v's X_1 and X_2 are sto. indep, we have

$$f_1(x_1 | x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = \frac{f_1(x_1) \cdot f_2(x_2)}{f_2(x_2)} = f_1(x_1)$$

and

$$f_2(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} = \frac{f_1(x_1)\, f_2(x_2)}{f_1(x_1)} = f_2(x_2)$$

An alternative definition for sto. indep is as follows:-

Let the r.v's X_1 and X_2 have the joint c.d.f $F(x_1, x_2)$ and the marginals c.d.f's $F_1(x_1)$ and $F_2(x_2)$ respectively. The r.v's X_1 and X_2 are said to be sto. indep iff $F(x_1, x_2) = F_1(x_1) \cdot F_2(x_2)$

Example (7) Back to Examples (1), (2), (3), and (4)

In Example (1), we have

(i) $f(x,y) = \frac{1}{21}(x+y)$, $x = 1,2,3$; $y = 1,2$

$\quad = 0$, e.w

and $f_1(x) = \frac{1}{21}(2x+3)$, $x = 1,2,3$, $f_2(y) = \frac{1}{7}(y+2)$, $y = 1,2$

$\quad = 0$, e.w $\qquad\qquad = 0$, e.w

$$f_1(x) \cdot f_2(y) = \frac{1}{217}(2x+3)(y+2) \neq \frac{1}{21}(x+y) = f(x,y)$$

and that implies the r.v's X and Y are sto. dep.

(ii) $f(x,y) = e^{-(x+y)}$, $0 < x < \infty$

$\quad = 0$, e.w

and $f_1(x) = e^{-x}$, $0 < x < \infty$, $f_2(y) = e^{-y}$, $0 < y < \infty$

$\quad = 0$, e.w $\qquad\qquad = 0$, e.w

$$f_1(x) \cdot f_2(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f(x,y) \Rightarrow X \text{ and } Y \text{ are sto. indep}$$

In Example (2), we have

$F(x,y)=\frac{1}{42}xy(x+y+2)$, $F_1(x)=\frac{1}{21}x(x+4)$, $F_2(y)=\frac{1}{14}y(y+5)$

$F_1(x)\cdot F_2(y)=\frac{1}{294}xy(x+4)(y+5)\neq\frac{1}{42}xy(x+y+2)=F(x,y)$

$\Rightarrow X$ and Y are sto. dep

(iii) $F(x,y)=(1-e^{-x})(1-e^{-y})$, $F_1(x)=1-e^{-x}$, $F_2(y)=1-e^{-y}$

$F_1(x)\cdot F_2(y)=(1-e^{-x})(1-e^{-y})=F(x,y)\Rightarrow X$ and Y are sto. indep.

In Example (3), we have

$f(x,y)=x+y$, $f_1(x)=x+\frac{1}{2}$, $f_2(y)=y+\frac{1}{2}$

$f_1(x)\cdot f_2(y)=(x+\frac{1}{2})(y+\frac{1}{2})\neq x+y=f(x,y)\Rightarrow X$ and Y are sto. dep.

In Example (4), we have

$f(x,y)=e^{-y}$, $f_1(x)=e^{-x}$, $f_2(y)=ye^{-y}$

$f_1(x)\cdot f_2(y)=e^{-x}\cdot ye^{-y}=ye^{-(x+y)}\neq e^{-y}=f(x,y)\Rightarrow X$ and Y are sto. dep.

<u>Theorem</u> (1) Let $f(x_1,x_2)$ be the joint p.d.f of r.v's X_1 and X_2. The r.v's X_1 and X_2 are sto. indep iff $f(x_1,x_2)$ can be written as a product of a non-negative function of x_1 alone and a non-negative function of x_2 alone, that is

$f(x_1,x_2)=g(x_1)\cdot h(x_2)$ where $g(x_1)>0$ and $h(x_2)>0$

proof

$\Rightarrow$ Let the r.v's X_1 and X_2 are sto. indep, then

$f(x_1,x_2)=f_1(x_1)\cdot f_2(x_2)$, where $f_1(x_1)$ is the marginal p.d.f of r.v X_1 which is a function of x_1 alone and $f_2(x_2)$ is the marginal p.d.f of X_2 which is a function of x_2 alone.

Thus, the condition $f(x_1,x_2)=g(x_1)h(x_2)$ is fulfilled.

$\Leftarrow$ Let $f(x_1,x_2)=g(x_1)h(x_2)$, where $g(x_1)>0$ is a function of x_1 alone and $h(x_2)>0$ is a function of x_2 alone.

Consider the cont. case

$$f_1(x_1) = \int_{x_2} f(x_1,x_2)\,dx_2 = \int_{x_2} g(x_1)\,h(x_2)\,dx_2 = g(x_1)\int_{x_2} h(x_2)\,dx_2 = c_1\,g(x_1)$$

and

$$f_2(x_2) = \int_{x_1} f(x_1,x_2)\,dx_1 = \int_{x_1} g(x_1)\,h(x_2)\,dx_1 = h(x_2)\int_{x_1} g(x_1)\,dx_1 = c_2\,h(x_2)$$

but $1 = \int_{x_1}\int_{x_2} f(x_1,x_2)\,dx_1\,dx_2 = \int_{x_1}\int_{x_2} g(x_1)\,h(x_2)\,dx_1\,dx_2$

$$= \left(\int_{x_1} g(x_1)\,dx_1\right)\cdot\left(\int_{x_2} h(x_2)\,dx_2\right) = c_1 c_2$$

$\therefore\ f(x_1,x_2) = g(x_1)\cdot h(x_2) = c_1 g(x_1)\cdot c_2 h(x_2) = f_1(x_1)\cdot f_2(x_2)$

$\therefore$ The r.v.'s X_1 and X_2 are sto. indep.

<u>Theorem (2)</u> If the r.v.'s X_1 and X_2 are sto. indep., then

(1) $Pr(a < X_1 < b,\ c < X_2 < d) = Pr(a < X_1 < b)\cdot Pr(c < X_2 < d)$

(2) $E[u(X_1)\cdot v(X_2)] = E[u(X_1)]\cdot E[v(X_2)]$, where $u(x_1)$ is a function of x_1 alone and $v(x_2)$ is a function of x_2 alone.

<u>proof</u>

Consider the cont. case

(1) $Pr(a < X_1 < b,\ c < X_2 < d) = \int_a^b\int_c^d f(x_1,x_2)\,dx_1\,dx_2$

$$= \int_a^b\int_c^d f_1(x_1)\cdot f_2(x_2)\,dx_1\,dx_2$$

$$= \left(\int_a^b f_1(x_1)\,dx_1\right)\cdot\left(\int_c^d f_2(x_2)\,dx_2\right)$$

$$= Pr(a < X_1 < b)\cdot Pr(c < X_2 < d)$$

(2) $E[u(X_1)\cdot v(X_2)] = \int_{x_1}\int_{x_2} u(x_1)\cdot v(x_2)\,f(x_1,x_2)\,dx_1\,dx_2$

$$= \int_{x_1}\int_{x_2} u(x_1)\cdot v(x_2)\,f_1(x_1)\cdot f_2(x_2)\,dx_1\,dx_2$$

$$= \left(\int_{x_1} u(x_1)\,f_1(x_1)\,dx_1\right)\cdot\left(\int_{x_2} v(x_2)\,f_2(x_2)\,dx_2\right)$$

$$= E[u(X_1)]\cdot E[v(X_2)]$$

If the r.v's X and Y are sto. indep, then

$E(XY) = E(X)\cdot E(Y) = \mu_1\mu_2$

(2) $cov(X,Y) = E(XY) - \mu_1\mu_2 = E(X)\cdot E(Y) - \mu_1\mu_2 = \mu_1\mu_2 - \mu_1\mu_2 = 0$

$\Rightarrow \rho = \rho(X,Y) = 0$, but the converse is not true. That is if $\rho = 0$ does not implies that X and Y are sto. indep.

Theorem (4) Let $f(x_1,x_2)$ be the joint p.d.f of r.v's X_1 and X_2 with marginals p.d.f's $f_1(x_1)$ and $f_2(x_2)$ and let $M(t_1,t_2)$ be the joint m.g.f of r.v's X_1 and X_2 with marginals m.g.f's $M_1(t_1)$ and $M_2(t_2)$. The r.v's X_1 and X_2 are sto. indep iff $M(t_1,t_2) = M_1(t_1)M_2(t_2)$

proof

$\Rightarrow$ Let the r.v's X_1 and X_2 be sto. indep., then

$$M(t_1,t_2) = E(e^{t_1X_1+t_2X_2}) = E(e^{t_1X_1}\cdot e^{t_2X_2}) = E(e^{t_1X_1})\cdot E(e^{t_2X_2})$$

by Theorem (3) part (2)

$$= M_1(t_1)\cdot M_2(t_2)$$

$\Leftarrow$ Let $M(t_1,t_2) = M_1(t_1)\cdot M_2(t_2)$ and consider the cont. case

$$M_1(t_1) = E(e^{t_1X_1}) = \int_{x_1} e^{t_1x_1} f_1(x_1)\,dx_1 \quad \text{and} \quad M_2(t_2) = E(e^{t_2X_2}) = \int_{x_2} e^{t_2x_2} f_2(x_2)\,dx_2$$

So

$$M_1(t_1)\cdot M_2(t_2) = \left(\int_{x_1} e^{t_1x_1} f_1(x_1)\,dx_1\right)\left(\int_{x_2} e^{t_2x_2} f_2(x_2)\,dx_2\right)$$

$$= \iint_{x_1\,x_2} e^{t_1x_1+t_2x_2} f_1(x_1) f_2(x_2)\,dx_1\,dx_2$$

But

$$M(t_1,t_2) = E(e^{t_1X_1+t_2X_2}) = \iint_{x_1\,x_2} e^{t_1x_1+t_2x_2} f(x_1,x_2)\,dx_1\,dx_2$$

and $M(t_1,t_2) = M_1(t_1)\cdot M_2(t_2)$, then

$$\iint_{x_1\,x_2} e^{t_1x_1+t_2x_2} f_1(x_1) f_2(x_2)\,dx_1\,dx_2 = \iint_{x_1\,x_2} e^{t_1x_1+t_2x_2} f(x_1,x_2)\,dx_1\,dx_2$$

that implies $f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$

$\therefore$ the r.v's X_1 and X_2 are sto. indep

Example(8) Back to Example(4), we have

$M(t_1,t_2) = (1-t_2)^{-1}(1-t_1-t_2)^{-1}$, $M_1(t_1) = (1-t_1)^{-1}$, $M_2(t_2) = (1-t_2)^{-1}$

$$M_1(t_1)\cdot M_2(t_2) = (1-t_1)^{-1}(1-t_2)^{-2} \neq (1-t_1)^{-1}(1-t_1-t_2)^{-1} = M(t_1,t_2)$$

$\therefore$ the r.v's X and Y are sto. dep.

Example(9) Let the joint m.g.f of r.v's X and Y be $M(t_1,t_2) = \dfrac{e^{\frac{t_1^2}{1-2t_2}}}{1-2t_2}$

(1) Find the marginal m.g.f of X and the marginal m.g.f of Y.

(2) Test wether or not X and Y are sto. indep.

(3) Find the correlation coefficient between X and Y.

Solution

(1) The marginal m.g.f of X and the marginal p.d.f Y are

$$M_1(t_1) = M(t_1,0) = e^{t_1^2}, \quad M_2(t_2) = M(0,t_2) = (1-2t_2)^{-1}$$

(2) $M_1(t_1)\cdot M_2(t_2) = \dfrac{e^{t_1^2}}{1-2t_2} \neq M(t_1,t_2) \Rightarrow X$ and Y are sto. dep.

(3) Let $\psi(t_1) = \ln M_1(t_1) = t_1^2 \Rightarrow \dfrac{\partial\psi(t_1)}{dt_1} = 2t_1 \Rightarrow \dfrac{\partial\psi(t_1)}{dt_1}\Big|_{t_1=0} = 0 = E(X)$

$$\frac{\partial^2\psi(t_1)}{dt_1^2} = 2 \Rightarrow \frac{\partial^2\psi(t_1)}{dt_1^2}\Big|_{t_1=0} = 2 = Var(X) = \sigma_1^2$$

Let $\psi(t_2) = \ln M_2(t_2) = -\ln(1-2t_2) \Rightarrow \dfrac{\partial\psi(t_2)}{\partial t_2} = \dfrac{2}{1-2t_2} = 2(1-2t_2)^{-1}$

$$\Rightarrow \frac{\partial\psi(t_2)}{\partial t_2}\Big|_{t_2=0} = 2 = E(Y) = \mu_2, \quad \frac{\partial^2\psi(t_2)}{\partial t_2^2} = 4(1-2t_2)^{-2}$$

$$\frac{\partial^2\psi(t_2)}{\partial t_2^2}\Big|_{t_2=0} = 4 = Var(Y) = \sigma_2^2$$

Let $\psi(t_1,t_2) = \ln M(t_1,t_2) = t_1^2(1-2t_2)^{-1} - \ln(1-2t_2)$

$$\frac{\partial\psi(t_1,t_2)}{\partial t_1} = 2t_1(1-2t_2)^{-1} \Rightarrow \frac{\partial^2\psi(t_1,t_2)}{\partial t_2\partial t_1} = 4t_1(1-2t_2)^{-2}$$

$$\frac{\partial^2\psi(t_1,t_2)}{\partial t_2\partial t_1}\Big|_{t_1=t_2=0} = 0 = cov(X,Y) \Rightarrow \rho = \frac{cov(X,Y)}{\sigma_1\sigma_2} = \frac{0}{2\sqrt{2}} = 0$$