Sampling Theory:-

Defn. (Statistic):-

A function of one or more r.v's which is not depend on any unknown parameter is called a statistic

For example:- If the r.v $X \sim N(\mu, \sigma^2)$, we may ~~ask whether~~ I ask If $Y = X - \mu$ is a statistic $\Rightarrow$ Y is not a statistic unless μ and σ are Known but if $Y = \sum_{i=1}^{n} X_i$ then Y is a statistic.

Defn. (Random Sample ≡ r.s)

Let $X_1, X_2, \ldots, X_n$ be sto. indep. r.v's each of which has the same p.d.f $f(x)$, that is $f_i(x_i) = f(x_i)$ $i = 1, 2, \ldots, n$, So the joint p.d.f of these r.v's
$$f(x_1, x_2, \ldots, x_n) = f(x_1) \cdot f(x_2) \cdots f(x_n) = \prod_{i=1}^{n} f(x_i)$$
~~then~~ the r.v's $X_1, X_2, \ldots, X_n$ are said to constitute a random sample of size n from a distn. whose p.d.f $f(x)$

Example (1):

Let X_1, X_2, X_3 be sto. indep. r.v's with $X_1 \sim Exp(\lambda)$, $X_2 \sim N(\mu, \sigma^2)$, and $X_3 \sim Be(\alpha, \beta)$. Then the joint p.d.f of X_1, X_2 and X_3 is

$$f(x_1, x_2, x_3) = \frac{1}{\lambda} e^{-\frac{x_1}{\lambda}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x_2 - \mu}{\sigma}\right)^2} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x_3^{\alpha-1} (1 - x_3)^{\beta-1}$$

$$= 0 \quad , \text{e.w}$$

$0 < x_1 < \infty$

$-\infty < x_2 < \infty$

$0 < x_3 < 1$

Example (2):

Let X_1, X_2, X_3 be sto. indep r.v's with $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2, 3$. Then the joint p.d.f of these r.v's is

$$f(x_1, x_2, x_3) = \prod_{i=1}^{3} f_i(x_i) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2} = (2\pi)^{-3/2} (\sigma_1 \sigma_2 \sigma_3)^{-1} e^{-\frac{1}{2} \sum \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2}$$

$-\infty < x_i < \infty, \; i = 1, 2, 3$

Example (3):

Let X_1, X_2, X_3 be sto. indep. r.v's with $X_i \sim N(\mu, \sigma^2)$ $i = 1, 2, 3$. Then the joint p.d.f of these r.v's is

$$f(x_1, x_2, x_3) = \prod_{i=1}^{3} f(x_i) = \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= (2\pi)^{-3/2} \sigma^{-3} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{3}(x_i-\mu)^2}$$

$$i = 1, 2, 3 \; , \; -\infty < x_i < \infty$$

Note: The r.v's X_1, X_2, X_3 in Example (2) are indep. but not represent a random sample while the r.v's X_1, X_2, X_3 in Example (3) represent a random sample of size 3 from $N(\mu, \sigma^2)$.

Example (4):

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from

a) $Ber(p)$ b) $Geo(p)$ c) $P(\lambda)$ d) $U(a,b)$

~~e) U(a,b)~~ $Exp(\lambda)$ f) $Be(\alpha, \beta)$ ~~g) U(a,b)~~

h) $N(0,1)$ i) $N(\mu, \sigma^2)$

Write the joint p.d.f in each case

Sample Mean and Sample Variance

Defn:

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from any distn. (Disc. or Cont.), we define

1) The statistic $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is called the sample mean

2) The statistic $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is called the sample Variance

(18)

Transformation (change) Variables p.d.f Technique :-

Continuous Type :-

Let $X_1, X_2, \ldots, X_n$ be n r.v's of cont. type defined on the n-dimensional spase with joint p.d.f $f(x_1, x_2, \ldots, x_n)$. Let the r.v's $Y_i = u_i(X_1, X_2, \ldots, X_n)$, $i = 1, 2, \ldots, k$ be functions of $X_1, X_2, \ldots, X_n$ $(K \leq n)$ suppose the joint p.d.f of $Y_1, Y_2, \ldots, Y_K$ is required.

If $K < n$, we introduce an additional new r.v's $Y_{k+1} = U_{k+1}(X_1, X_2, \ldots, X_n)$, $Y_{k+2} = U_{k+2}(X_1, X_2, \ldots, X_n), \ldots,$ $Y_n = U_n(X_1, X_2, \ldots, X_n)$ so that the functions $y_i = u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$ define one-to-one transformation that maps the space $\mathcal{A}$ of $\{X_i\}$ onto the space $\mathcal{B}$ of $\{Y_i\}$ with inverse transfors $x_i = w_i(y_1, y_2, \ldots, y_n)$, $i = 1, 2, \ldots, n$ and the Jacobian nxn determinant of the 1^{st} partial derivatives

$$J = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \neq 0$$

Then the joint p.d.f of $Y_1, Y_2, \ldots, Y_n$ is

$$g(y_1, y_2, \ldots, y_n) = f[w_1(y_1, y_2, \ldots, y_n), w_2(y_1, y_2, \ldots, y_n), \ldots, w_n(y_1, y_2, \ldots, y_n)] \, |J|$$

and the joint p.d.f of $Y_1, Y_2, \ldots, Y_K$ is

$$g^*(y_1, y_2, \ldots, y_K) = \int_{y_{k+2}} \int_{y_{k+1}} \cdots \int_{y_n} g(y_1, y_2, \ldots, y_n) \, dy_n \ldots dy_{k+1} \, dy_{k+2}$$

Example(1):
Let the r.v X has p.d.f $f(x) = 3x^2 , \quad 0 < x < 1$
$= 0 , \quad e.w$

Determine the p.d.f of r.v $Y = X^3$ and evaluate $P_r(\frac{1}{2} < Y < \frac{3}{4})$

Sol:

~~Let the r.v X has p.d.f~~

The function $y = x^3$: $A = \{x : 0 < x < 1\} \rightarrow B = \{y : 0 < y < 1\}$
with inverse $x = y^{1/3}$ and $J = \frac{dx}{dy} = \frac{1}{3} y^{-2/3}$. Then the p.d.f of Y is

$$g(y) = f(y^{1/3}) |J| = 3 (y^{1/3})^2 \frac{1}{3} y^{-2/3} = 1$$

$$\therefore g(y) = 1 , \quad 0 < y < 1$$
$$= 0 , \quad e.w$$

Example (2):
Let the r.v $X \sim U(0,1)$. Find and name the distn. of r.v $Y = -2 \ln X$.

Sol:
The function $y = -2 \ln x$: $A = \{x : 0 < x < 1\} \rightarrow B = \{y : 0 < y < \infty\}$
with inverse $x = e^{-\frac{1}{2}y}$ and

$$J = \frac{dx}{dy} = -\frac{1}{2} e^{-\frac{1}{2}y}$$. Then the p.d.f of Y is

$$g(y) = f(e^{-\frac{1}{2}y}) |J| = (1) \frac{1}{2} e^{-\frac{1}{2}y} = \frac{1}{2} e^{-\frac{y}{2}} , \quad 0 < y < \infty$$
$$= 0 , \quad e.w$$

That is $Y \sim Exp(2)$

Example (3)

Let the r.v X has p.d.f $f(x) = \frac{1}{\pi}(1+x^2)^{-1}$, $-\infty < x < \infty$

$= 0$, e.w

Find the p.d.f of r.v $Y = X^2$

Sol.:

The function $y = x^2$: $\mathcal{A} = \{x : -\infty < x < \infty\} \xrightarrow[\text{on to}]{\text{not } (1-1)}$ $\mathcal{B} = \{y : 0 < y < \infty\}$. So we write the S.S $\mathcal{A}$ as a union of two mutually exclusive subspaces, say $\mathcal{A}_1$ and $\mathcal{A}_2$.

That is $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \phi$ where

$\mathcal{A}_1 = \{x : -\infty < x < 0\}$ and $\mathcal{A}_2 = \{x : 0 \le x < \infty\}$

Now the function $y = x^2$ define (1-1) transformation that maps of $\mathcal{A}_1$ and $\mathcal{A}_2$ onto $\mathcal{B}$ with inverses

In $\mathcal{A}_1$

$x = -\sqrt{y}$ and $J_1 = \frac{dx}{dy} = \frac{-1}{2\sqrt{y}}$

In $\mathcal{A}_2$

$x = \sqrt{y}$ and $J_2 = \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

then the p.d.f of Y is

$$g(y) = f(-\sqrt{y})\,|J_1| + f(\sqrt{y})\,|J_2|$$

$$= \frac{1}{\pi}(1+y)^{-1}\frac{1}{2\sqrt{y}} + \frac{1}{\pi}(1+y)^{-1}\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\pi}\,\frac{1}{(1+y)\sqrt{y}}, \quad 0 < y < \infty$$

$$= 0, \quad \text{e.w}$$

Example (4):

Let X_1, X_2 be a r.s of size 2 from $N(0,1)$. Define the r.v's $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint p.d.f of Y_1 and Y_2 and show that they are sto. indep. Name the distn. of Y_1 and of Y_2

Sol.:

The joint p.d.f of X_1 and X_2 is

$$f(x_1, x_2) = f(x_1)\cdot f(x_2) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(x_1^2 + x_2^2)\right], \ -\infty < x_i < \infty$$

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The functions $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$: $A = \{(x_1, x_2) : -\infty < x_i < \infty; i = 1, 2\} \to B = \{(y_1, y_2) : -\infty < y_i < \infty, i = 1, 2\}$
with inverses $x_1 = \frac{1}{2}(y_1 + y_2)$, $x_2 = \frac{1}{2}(y_1 - y_2)$
and

$$J = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$x_1^2 + x_2^2 = \frac{1}{4}(y_1 + y_2)^2 + \frac{1}{4}(y_1 - y_2)^2$$

$$= \frac{1}{4}\left[y_1^2 + 2y_1y_2 + y_2^2 + y_1^2 - 2y_1y_2 + y_2^2\right]$$

$$= \frac{1}{4}(2y_1^2 + 2y_2^2) = \frac{1}{2}(y_1^2 + y_2^2)$$. Then the joint p.d.f of Y_1 and Y_2 is

$$g(y_1, y_2) = f\left[\tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(y_1 - y_2)\right] |J|$$

$$= \frac{1}{4\pi} e^{-\frac{1}{4}(y_1^2 + y_2^2)}, \quad -\infty < y_i < \infty, \ i = 1, 2$$

$= 0$, e.w

The marginal p.d.f of Y_1 is $g_1(y_1) = \int_{y_2} g(y_1, y_2)\, dy_2$

$$g_1(y_1) = \frac{1}{4} e^{-\frac{1}{4}y_1^2} \int_{-\infty}^{\infty} e^{-\frac{1}{4}y_2^2}\, dy_2$$

$$= \frac{e^{-\frac{1}{4}y_1^2}}{4\pi} \sqrt{2\pi}\sqrt{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_2^2}{2}}\, dy_2$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_1^2}{2}}, \quad -\infty < y_1 < \infty$$

$= 0$, e.w

That is $Y_1 \sim N(0, 2)$

Similarly the p.d.f of Y_2 is $g_2(y_2) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_2^2}{2}}, \ -\infty < y_2 < \infty$

$$g_1(y_1) \cdot g_2(y_2) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_2^2}{2}} = \frac{1}{4\pi} e^{-\frac{1}{4}(y_1^2 + y_2^2)} = g(y_1, y_2)$$

Therefore Y_1 and Y_2 are sto. indep.

Example (6):

Let X_1 and X_2 be two indep. r.v's with $X_1 \sim N(0,1)$ and $X_2 \sim \chi^2_{(2)}$. Find the distn. of r.v $Y = \frac{X_1}{\sqrt{\frac{X_2}{2}}}$

Sol.:

$$X_1 \sim N(0,1) \Rightarrow f_1(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} \quad , -\infty < x_1 < \infty$$
$$= 0 \quad , \text{e.w}$$

$$X \sim \chi^2_{(2)} \Rightarrow f_2(x_2) = \frac{1}{2} e^{-\frac{x_2}{2}} \quad , 0 < x_2 < \infty$$
$$= 0 \quad , \text{e.w}$$

Since X_1 and X_2 are sto. indep., then the joint p.d.f of X_1 and X_2 is

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(x_1^2 + x_2)} \quad , -\infty < x_1 < \infty, \; 0 < x_2 < \infty$$
$$= 0 \quad , \text{e.w}$$

with $Y = \frac{X_1}{\sqrt{\frac{X_2}{2}}}$, set $Z = X_2$

The functions $y = \frac{x_1}{\sqrt{\frac{x_2}{2}}}$, $z = x_2$: $\mathcal{A} = \{(x_1, x_2) : -\infty < x_1 < \infty, \; 0 < x_2 < \infty\}$

$\longrightarrow \mathcal{B} = \{(y, z) : -\infty < y < \infty, \; 0 < z < \infty\}$ with inverses

$x_1 = \frac{y\sqrt{z}}{\sqrt{2}}$, $x_2 = z$ and

$$J = \frac{\partial(x_1, x_2)}{\partial(y, z)} = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{z}}{\sqrt{2}} & \frac{y}{2\sqrt{2}\sqrt{z}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{z}{2}}$$

Then the joint p.d.f of Y and Z is

$$g(y, z) = f\left(\frac{y\sqrt{z}}{\sqrt{2}}, z\right) |J|$$
$$= \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{zy^2}{2} + z\right)} \cdot \sqrt{\frac{z}{2}}$$

$$= \frac{1}{4\sqrt{\pi}} z^{\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{y^2}{2}+1\right)z} \quad , -\infty < y < \infty , \; 0 < z < \infty$$

$$= 0 \quad , \text{e.w}$$

The marginal p.d.f of Y is

$$g_1(y) = \int_z g(y,z)\,dz = \frac{1}{4\sqrt{\pi}} \int_0^\infty z^{\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{y^2}{2}+1\right)z}\,dz$$

set $w = \left(\frac{y^2}{2}+1\right)z \Rightarrow dw = \left(\frac{y^2}{2}+1\right)dz$

$$g_1(y) = \frac{1}{4\sqrt{\pi}} \int_0^\infty \frac{w^{1/2}}{\left(\frac{y^2}{2}+1\right)^{1/2}} e^{-\frac{w}{2}} \cdot \frac{dw}{\left(\frac{y^2}{2}+1\right)}$$

$$= \frac{1}{4\sqrt{\pi}} \left(\frac{y^2}{2}+1\right)^{-3/2} \int_0^\infty w^{1/2} e^{-\frac{w}{2}}\,dw$$

$$= \frac{1}{4\sqrt{\pi}} \left(\frac{y^2}{2}+1\right)^{-3/2} \Gamma\left(\frac{3}{2}\right) \cdot 2^{3/2} \int_0^{\infty} \frac{1}{\Gamma\left(\frac{3}{2}\right) 2^{3/2}} w^{\frac{3}{2}-1} e^{-\frac{w}{2}}\,dw$$

$$= \frac{1}{4\sqrt{\pi}} \left(\frac{y^2}{2}+1\right)^{-3/2} \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \cdot 2\sqrt{2}$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{y^2}{2}+1\right)^{-3/2} \quad , -\infty < y < \infty$$

$$= 0 \quad , \text{e.w}$$

The t-distn. (student-distn.)

In 1908 Gosset derive the equation of t-distn. for small samples. He publised his work secretly under the name "student".

Theorem (Gosset Theorem):-

Let X and Y be sto. indep. r.v's with $X \sim N(0,1)$ and $Y \sim \chi^2_{(r)}$. Then the r.v $T = \frac{X}{\sqrt{\frac{Y}{r}}}$ has p.d.f

$$g(t) = \frac{\Gamma\left(\frac{r+2}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{t^2}{r}\right)^{-\frac{r+1}{2}}, \quad -\infty < t < \infty,\ r > 0$$

and T is said to have a t-distn. with r degress of freedom and denoted by $T \sim t(r)$

Proof:-

Given $X \sim N(0,1)$ then the p.d.f of X is

$$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty$$

$= 0$

also $Y \sim \chi^2_{(r)}$ then the p.d.f of Y is

$$f_2(y) = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}}\, y^{r/2-1}\, e^{-\frac{y}{2}}, \quad 0 < y < \infty$$

$$= 0 \quad , \text{e.w}$$

Since X and Y are sto. indep. then the joint p.d.f of X and Y is

$$f(x,y) = f_1(x) f_2(y) = \frac{1}{\sqrt{2\pi}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}}\, y^{r/2-1}\, e^{-\frac{1}{2}(x^2+y)}, \quad -\infty < x < \infty,\ 0 < y < \infty$$

$$= 0 \quad , \text{e.w}$$

We have the transformation $T = \frac{X}{\sqrt{\frac{Y}{r}}}$, Let $W = Y$

Now, the function $t = \frac{x}{\sqrt{\frac{y}{r}}}$ and $w = y$ define a one-to-one transformation that maps the space $A=\{(x,y): -\infty<x<\infty, 0<y<\infty\}$ onto the space $B=\{(t,w): -\infty<t<\infty, 0<w<\infty\}$ with inverse transform

$x=t\frac{\sqrt{w}}{\sqrt{r}}$, $y=w$ and the Jacobian of transformation is

$$\cancel{x=t\frac{\sqrt{w}}{\sqrt{r}},\ y=w}$$

$$J=\frac{\partial(x,y)}{\partial(t,w)}=\begin{vmatrix}\frac{\partial x}{\partial t} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w}\end{vmatrix}=\begin{vmatrix}\frac{\sqrt{w}}{\sqrt{r}} & \frac{t}{2\sqrt{r}\sqrt{w}}\\ 0 & 1\end{vmatrix}=\frac{\sqrt{w}}{\sqrt{r}}$$

Then the joint p.d.f of T and w is given by.

$$g(t,w)=f\left(t\frac{\sqrt{w}}{\sqrt{r}},w\right)|J|=\frac{1}{\sqrt{2\pi}\,\Gamma\left(\frac{r}{2}\right)2^{r/2}}w^{r/2-1}e^{-\frac{1}{2}\left(\frac{wt^2}{r}+w\right)}\frac{\sqrt{w}}{\sqrt{r}}$$

$$=\frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right)2^{r/2}}w^{\frac{r+1}{2}-1}e^{-\frac{1}{2}\left(1+\frac{t^2}{r}\right)w},\quad -\infty<t<\infty,\ 0<w<\infty$$

$$=0 \quad , \text{e.w}$$

The marginal p.d.f of T is

$$g_1(t)=\int_w g(t,w)\,dw=\frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right)2^{r/2}}\int_{w=0}^{\infty}w^{\frac{r+1}{2}-1}e^{-\frac{1}{2}\left(1+\frac{t^2}{r}\right)w}\,dw$$

Let $z=w\left(1+\frac{t^2}{r}\right) \Rightarrow dz=\left(1+\frac{t^2}{r}\right)dw$

$$g_1(t)=\frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right)2^{r/2}}\int_{z=0}^{\infty}\left(\frac{z}{1+\frac{t^2}{r}}\right)^{\frac{r+1}{2}-1}e^{-\frac{z}{2}}\left(1+\frac{t^2}{r}\right)^{-1}dz$$

$$=\frac{1}{\sqrt{2\pi r}\,\Gamma\left(\frac{r}{2}\right)2^{r/2}}\cdot\frac{\Gamma\left(\frac{r+1}{2}\right)2^{\frac{r+1}{2}}}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}-1+1}}\int_0^{\infty}\frac{1}{\Gamma\left(\frac{r+1}{2}\right)2^{\frac{r+1}{2}}}z^{\frac{r+1}{2}-1}e^{-\frac{z}{2}}\,dz$$

$$=\frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{t^2}{r}\right)^{-\frac{r+1}{2}},\quad -\infty<t<\infty$$

The c.d.f of T is

$G_1(t) = Pr(T \leq t) = \int_{-\infty}^{t} g_1(s)\,ds$. This integral dose not have a simple form.

Some approximate value of $Pr(T \leq t)$ for selected values of t and r are given in table IV of Appendix B.

The m.g.f of t_distn does not exist. To find the mean and variance of T can be found as follows:

We have $T = \frac{X}{\sqrt{\frac{Y}{r}}}$ where X and Y are sto. indep. with $X \sim N(0,1)$ and $Y \sim \chi^2_{(r)}$

$$E(T) = E\left(X \cdot \frac{1}{\sqrt{\frac{Y}{r}}}\right) = E(X) \cdot E\left(\frac{1}{\sqrt{\frac{Y}{r}}}\right) = 0$$

$$E(T^2) = E\left(X^2 \cdot \frac{r}{Y}\right) = E(X^2) \cdot r E\left(\frac{1}{Y}\right) = r E\left(\frac{1}{Y}\right)$$

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} y^{\frac{r}{2}-1} e^{-\frac{y}{2}}\, dy$$

$$= \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} \Gamma\left(\frac{r-2}{2}\right) 2^{\frac{r-2}{2}} \int_0^\infty \frac{1}{\Gamma(\frac{r-2}{2})\, 2^{\frac{r-2}{2}}} y^{\frac{r-2}{2}-1} e^{-\frac{y}{2}}\, dy$$

$$= \frac{\Gamma(\frac{r-2}{2})}{2\Gamma(\frac{r}{2})} = \frac{\Gamma(\frac{r}{2}-1)}{2\Gamma(\frac{r}{2})} = \frac{\Gamma(\frac{r}{2}-1)}{2(\frac{r}{2}-1)\Gamma(\frac{r}{2}-1)} = \frac{1}{r-2}, \quad r > 2$$

$$Var(T) = E(T^2) - \{E(T)\}^2 = \frac{r}{r-2} - 0 = \frac{r}{r-2}, \quad r \neq 2$$

Example (1)

Let the r.v $X \sim t(10)$. Find $Pr(|X| > 2.228)$

Sol.:

$$Pr(|X| > 2.228) = 1 - Pr(|X| \leq 2.228)$$
$$= 1 - Pr(-2.228 \leq X \leq 2.228)$$
$$= 1 - [2 Pr(X \leq 2.228) - 1]$$
$$= 2 - 2 Pr(X \leq 2.228)$$
$$= 2[1 - Pr(X \leq 2.228)] = 2(1 - 0.975)$$
$$= 2(0.025) = 0.05$$

The F-Distn. (Fisher Theorem)

Let X and Y be sto. indep. r.v's with $X \sim \chi^2_{(r)}$ and $Y \sim \chi^2_{(s)}$. Then the r.v $F = \frac{X/r}{Y/s} = \frac{sX}{rY}$ has p.d.f

$$g(f) = \frac{\Gamma(\frac{r+s}{2}) (\frac{r}{s})^{r/2}}{\Gamma(\frac{r}{2}) \Gamma(\frac{s}{2})} \cdot \frac{f^{r/2 - 1}}{(1 + \frac{r}{s} f)^{\frac{r+s}{2}}}, \quad 0 < f < \infty$$
$$= 0 \quad , \text{e.w}$$

and F is said to have F-distn with (r, s) dof's and denoted by $F \sim F(r, s)$

Proof:

Since X and Y are indep. r.v's, then the joint p.d.f is

$$f(x, y) = \frac{1}{\Gamma(\frac{r}{2}) 2^{r/2}} x^{\frac{r}{2} - 1} e^{-\frac{x}{2}} \cdot \frac{1}{\Gamma(\frac{s}{2}) 2^{\frac{s}{2}}} y^{\frac{s}{2} - 1} e^{-\frac{y}{2}}$$

$$= \frac{1}{\Gamma(\frac{r}{2}) \Gamma(\frac{s}{2}) 2^{\frac{r+s}{2}}} x^{\frac{r}{2} - 1} y^{\frac{s}{2} - 1} e^{-\frac{1}{2}(x+y)}, \quad 0 < x < \infty, 0 < y < \infty$$
$$= 0 \quad , \text{e.w}$$

with $F = \frac{sX}{rY}$, Set $W = Y$

The functions $f = \frac{sx}{ry}$, $w = y$ $\mathcal{A} = \{(x, y) : 0 < x < \infty, 0 < y < \infty\}$ $\xrightarrow{rc}$

$\longrightarrow \mathcal{B} = \{(f, w) : 0 < f < \infty, 0 < w < \infty\}$ with inverses

$x = \frac{r}{s} f w$, $y = w$ and

$$J = \frac{\partial(x,y)}{\partial(f,w)} = \begin{vmatrix} \frac{\partial x}{\partial f} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial f} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{r}{s} w & \frac{r}{s} f \\ 0 & 1 \end{vmatrix} = \frac{r}{s} w$$

Then the joint p.d.f of F and W is

$$g(f,w) = f\left(\frac{r}{s} f w, w\right)|J| = \frac{1}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right) 2^{\frac{r+s}{2}}} \left(\frac{r}{s} f w\right)^{\frac{r}{2}-1} w^{\frac{s}{2}-1} e^{-\frac{1}{2}\left(\frac{r}{s} f + 1\right) w} \cdot \frac{r}{s} w \quad , 0 < f < \infty , 0 < w < \infty$$

$$= 0 \quad , \text{e.w}$$

The marginal p.d.f of F is

$$g_1(f) = \int_w g(f,w)\, dw = \frac{\left(\frac{r}{s}\right)^{r/2} f^{\frac{r}{2}-1}}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right) 2^{\frac{r+s}{2}}} \int_{w=0}^{\infty} w^{\frac{r+s}{2}-1} e^{-\frac{1}{2}\left(\frac{r}{s} f + 1\right) w}\, dw$$

Set $Z = \left(\frac{r}{s} f + 1\right) w \Rightarrow dz = \left(\frac{r}{s} f + 1\right) dw$

$$g_1(f) = \frac{\left(\frac{r}{s}\right)^{\frac{r}{2}} f^{\frac{r}{2}-1}}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right) 2^{\frac{r+s}{2}}} \int_{z=0}^{\infty} \left(\frac{z}{\frac{r}{s} f + 1}\right)^{\frac{r+s}{2}-1} e^{-\frac{z}{2}} \frac{dz}{\left(\frac{r}{s} f + 1\right)}$$

$$= \frac{\left(\frac{r}{s}\right)^{r/2} f^{\frac{r}{2}-1} \Gamma\left(\frac{r+s}{2}\right) 2^{\frac{r+s}{2}}}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right) 2^{\frac{r+s}{2}} \left(\frac{r}{s} f + 1\right)^{\frac{r+s}{2}}} \int_{z=0}^{\infty} \frac{Z^{\frac{r+s}{2}-1}}{\Gamma\left(\frac{r+s}{2}\right) 2^{\frac{r+s}{2}}} e^{-\frac{z}{2}}\, dz$$

$$= \frac{\Gamma\left(\frac{r+s}{2}\right)\left(\frac{r}{s}\right)^{r/2}}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{s}{2}\right)} \frac{f^{r/2 - 1}}{\left(1 + \frac{r}{s} f\right)^{\frac{r+s}{2}}} \quad , 0 < f < \infty$$

$$= 0 \quad , \text{e.w}$$

The c.d.f of F is $G_1(f) = \int_{-\infty}^{f} g(w)\, dw = Pr(F \leq f)$. This integral does not have simple form. Some approximate value of $Pr(F \leq f)$ are given in table V of appendix A-7 for selected value of r, s and f

Example (2)
If the r.v $X \sim F(r,s)$. What is the distn of r.v $Y = \frac{1}{X}$

Sol:

Since $X \sim F(r,s)$ that mean there are two indep. r.v's say U and V such that $U \sim \chi^2_{(r)}$ and $V \sim \chi^2_{(s)}$ and

$$X = \frac{U/r}{V/s} \sim F(r,s)$$

Now

$$Y = \frac{1}{X} = \frac{1}{\frac{U/r}{V/s}} = \frac{V/s}{U/r} \sim F(s,r)$$

Important Note:

If $X \sim F(r,s)$ then $Y = \frac{1}{X} \sim F(s,r)$

$$Pr(X \leq a) = Pr\left(\frac{1}{X} \geq \frac{1}{a}\right) = 1 - Pr\left(\frac{1}{X} \leq \frac{1}{a}\right) = 1 - Pr\left(Y \leq \frac{1}{a}\right)$$

Example (3)
Let the r.v $X \sim F(5,10)$. Find the values of a and b so that $Pr(X \leq a) = 0.05$ and $Pr(X \leq b) = 0.95$

Sol:

$Pr(X \leq b) = 0.95$. From F-distn. table with $r=5$, $s=10$ we have $b = 3.33$

$$0.05 = Pr(X \leq a) = Pr\left(\frac{1}{X} \geq \frac{1}{a}\right) = 1 - Pr\left(\frac{1}{X} \leq \frac{1}{a}\right)$$
$$= 1 - Pr\left(Y \leq \frac{1}{a}\right)$$

where $Y \sim F(10,5) \Rightarrow Pr\left(Y \leq \frac{1}{a}\right) = 1 - 0.05 = 0.95$

From F-distn table with $r=10$, $s=5$, we have $\frac{1}{a} = 0.474$

$\Rightarrow a = 2.11$

The Distribution of $\bar{X}$ and $\frac{(n-1)S^2}{\sigma^2}$ (Normal Case)

Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from $N(\mu, \sigma^2)$ and suppose that we wish to find the distn. of $\bar{X}$ and $\frac{(n-1)S^2}{\sigma^2}$

Distn. of $\bar{X}$:-

For Normal case, the sample mean $\bar{X} = \frac{1}{n}\sum X_i \sim N(\mu, \frac{\sigma^2}{n})$

Example (1)

Let $X_1, X_2, \ldots, X_n$ be a r.s of size $n=25$ from $N(75, 100)$.
Find $Pr(71 < \bar{X} < 79)$

Sol:

We have $n=25$, $\mu=75$, $\sigma^2=100$

each $X_i \sim N(75, 100) \Rightarrow \bar{X} \sim N(75, \frac{100}{25}) \equiv N(75, 4)$

$$\Rightarrow Y = \frac{\bar{X}-75}{2} \sim N(0,1)$$

$$Pr(71 < \bar{X} < 79) = Pr\left(\frac{71-75}{2} < \frac{\bar{X}-75}{2} < \frac{79-75}{2}\right) = Pr(-2 < Y < 2)$$
$$= 2Pr(Y<2) - 1 = 2(0.977) - 1 = 0.954$$

Distn. of $\frac{(n-1)S^2}{\sigma^2}$:

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum\left[(X_i-\bar{X}) + (\bar{X}-\mu)\right]^2 = \sum_{i=1}^{n}\left[(X_i-\bar{X})^2 + 2(X_i-\bar{X})(\bar{X}-\mu) + (\bar{X}-\mu)^2\right]$$
$$= \sum(X_i-\bar{X})^2 + 2(\bar{X}-\mu)\sum(X_i-\bar{X}) + n(\bar{X}-\mu)^2$$
$$= \sum(X_i-\bar{X})^2 + 2(\bar{X}-\mu)(n\bar{X}-n\bar{X}) + n(\bar{X}-\mu)^2$$
$$= \sum(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2 = (n-1)S^2 + n(\bar{X}-\mu)^2$$

$$\Rightarrow (n-1)S^2 = \sum(X_i-\mu)^2 - n(\bar{X}-\mu)^2$$

$$\frac{(n-1)S^2}{\sigma^2} = \sum\left(\frac{X_i-\mu}{\sigma}\right)^2 - n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

Since each $X_i \sim N(\mu, \sigma^2) \Rightarrow \frac{x_i - \mu}{\sigma} \sim N(0,1)$

$\Rightarrow \left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)} \Rightarrow \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

also $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

$\Rightarrow n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$

Note: For normal case only we shall accept that $\bar{X}$ and S^2 are sto. indep.

Then $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Example (2)
Let $X_1, X_2, \ldots, X_n$ be a r.s of size $n=6$ from $N(\mu, 10)$
Find $P_r(2.3 < S^2 < 22.2)$
Sol:

$Y = \frac{(n-1)S^2}{\sigma^2} = \frac{5S^2}{10} = \frac{S^2}{2} \sim \chi^2_{(5)}$

$P_r(2.3 < S^2 < 22.2) = P_r\left(\frac{2.3}{2} < \frac{S^2}{2} < \frac{22.2}{2}\right)$
$= P_r(1.15 < Y < 11.1)$
$= P_r(Y < 11.1) - P_r(Y < 1.15)$
$= 0.95 - 0.05 = 0.9$

Mean and Variance of $\bar{X}$ and S^2 (Any distn.)
Let $X_1, X_2, \ldots, X_n$ be a r.s of size n from any distn. (discrete or continuous) having mean μ and variance σ^2

Mean and Variance of $\bar{X}$

$\bar{X} = \frac{1}{n}\sum X_i = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$

$E(\bar{X}) = \frac{1}{n}(EX_1 + EX_2 + \cdots + EX_n)$
$= \frac{1}{n}(\mu + \mu + \mu + \cdots + \mu) = \frac{1}{n}(n\mu) = \mu$

$$Var(\bar{X}) = \frac{1}{n^2}\left[Var(X_1) + Var(X_2) + \dots + Var(X_n)\right)$$

$$= \frac{1}{n^2}(\sigma^2 + \sigma^2 + \dots + \sigma^2) = \frac{n}{n^2}\sigma^2 = \frac{1}{n}\sigma^2$$

Mean and variance of S^2

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \Rightarrow (n-1)S^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

$$(n-1)E(S^2) = \sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2) = \sum(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= n(\sigma^2 + \mu^2) - \sigma^2 - n\mu^2 = n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2$$

$$= n\sigma^2 - \sigma^2 = (n-1)\sigma^2$$

So

$E(S^2) = \sigma^2$

We can show that

$Var(S^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)$ where $\mu_r = E[(X-\mu)^r]$

For examples

1.) If the sample from $G(\alpha, \beta)$, then $\mu = \alpha\beta$, $\sigma^2 = \alpha\beta^2$

So $E(\bar{X}) = \mu = \alpha\beta$ and $Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{\alpha\beta^2}{n}$

and

$E(S^2) = \sigma^2 = \alpha\beta^2$

$Var(S^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)$ where $\mu_4 = E[(X-\mu)^4]$

$$\mu_4 = E(X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4)$$

$$= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4$$

$$= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2 - 3\mu^4$$