Confidence Interval For The Mean μ

Normal Case

Let $X_1, X_2, \dots, X_n$ be a r.s of size n from $N(\mu, \sigma^2)$. Suppose we require C.I for the mean μ with prob. $1-\alpha$.

There are two cases :

Case (i) When σ^2 is known

For normal case, we know that the sample mean $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$\Rightarrow Z = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$. Hence, we can find from $N(0,1)$ distn table two no.'s, say $\pm z_{1-\frac{\alpha}{2}}$ suchthat $Pr(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}) = 1-\alpha$

Now

$$-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}} \equiv -z_{1-\frac{\alpha}{2}} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < z_{1-\frac{\alpha}{2}} \equiv \bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$$

$\therefore$ The $100(1-\alpha)\%$ C.I for μ is $\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

Note In practice, if $x_1, x_2, \dots, x_n$ are the observed values of r.v's $X_1, X_2, \dots, X_n$, then the $100(1-\alpha)\%$ C.I for μ is $\left(\bar{x} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{x} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$ where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Example (4) Given the sample 3.3, −0.3, −0.6, −0.9 from $N(\mu, 9)$. Find the 90% C.I for μ.

Solution we have $n = 4$, $\sigma = 3$, $\bar{x} = \frac{1}{4}(3.3-0.3-0.6-0.9) = 0.375$

$1-\alpha = 0.9 \Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05 \Rightarrow 1-\frac{\alpha}{2} = 0.95$

From $N(0,1)$ distn table, we have $z_{1-\frac{\alpha}{2}} = z_{0.95} = 1.645$

$\therefore$ The 90% C.I for μ is $\left(\bar{x} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{x} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

$$\equiv \left(0.375 - (\tfrac{3}{2})(1.645), 0.375 + (\tfrac{3}{2})(1.645)\right) \equiv (-2.093, 2.843)$$

Example (5) An observed sample of size 40 from $N(\mu, 10)$ yield $\bar{x} = 7.164$. Find the 80% C.I for μ.

Solution we have $n = 40$, $\sigma = \sqrt{10}$ $\bar{x} = 7.164$

$1-\alpha = 0.8 \Rightarrow \alpha = 0.2 \Rightarrow \frac{\alpha}{2} = 0.1 \Rightarrow 1-\frac{\alpha}{2} = 0.9$

From $N(0,1)$ distn table, we have $z_{1-\frac{\alpha}{2}} \equiv z_{0.9} \equiv 1.282$

The 80% C.I for μ is $\left(\bar{x} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{x} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

$$\equiv \left(7.164 - \frac{\sqrt{10}}{\sqrt{40}}(1.282), 7.164 + \frac{\sqrt{10}}{\sqrt{40}}(1.282)\right) \equiv (6.523, 7.805)$$

Case (ii) When σ^2 is unknown

For normal case, we know the following

(1) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

(2) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ (3). $\bar{X}$ and S^2 are sto. indep.

Then the r.v $T = \dfrac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{\sqrt{\dfrac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}} = \frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t(n-1)$

Hence, we can find from t-distn table with $(n-1)$ d.o.f two no.s, say $\pm t_{1-\frac{\alpha}{2}}$ suchthat $P_r(-t_{1-\frac{\alpha}{2}} < T < t_{1-\frac{\alpha}{2}}) = 1-\alpha$

Now $-t_{1-\frac{\alpha}{2}} < T < t_{1-\frac{\alpha}{2}} \equiv -t_{1-\frac{\alpha}{2}} < \frac{\sqrt{n}(\bar{X}-\mu)}{S} < t_{1-\frac{\alpha}{2}} \equiv \bar{X} - \frac{S}{\sqrt{n}} t_{1-\frac{\alpha}{2}} < \mu < \bar{X} + \frac{S}{\sqrt{n}} t_{1-\frac{\alpha}{2}}$

$\therefore$ The $100(1-\alpha)\%$ C.I for μ is $\left(\bar{X} - \frac{S}{\sqrt{n}} t_{1-\frac{\alpha}{2}}, \bar{X} + \frac{S}{\sqrt{n}} t_{1-\frac{\alpha}{2}}\right)$

Note In practice, if $x_1, x_2, \dots, x_n$ are the observed values of r.v's $X_1, X_2, \dots, X_n$, then the $100(1-\alpha)\%$ C.I for μ is $\left(\bar{x} - \frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2}}, \bar{x} + \frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2}}\right)$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right]$

Example (6) Back to Example (4) and suppose that the sample 3.3, −0.3, −0.6, −0.9 from $N(\mu, \sigma^2)$. What is the 90% C.I for μ.

Solution We have $n = 4$, σ unknown, $\bar{x} = 0.375$

$$s^2 = \frac{1}{3}\left[(3.3)^2 + (-0.3)^2 + (-0.6)^2 + (-0.9)^2 - (4)(0.375)^2\right]$$

$$= \frac{1}{3}(12.15 - 0.5625) = 3.8625 \Rightarrow s = 1.965$$

$1-\alpha=0.9 \Rightarrow \alpha=0.1 \Rightarrow \frac{\alpha}{2}=0.05 \Rightarrow 1-\frac{\alpha}{2}=0.95$

From t-distn table with 3 dof, we have $t_{1-\frac{\alpha}{2}} \equiv t_{0.95} \equiv 2.353$

The 90% c.I for μ is $\left(\bar{x}-\frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2}}, \bar{x}+\frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2}}\right)$

$\equiv\left(0.375-\frac{1.962}{2}(2.353), 0.375+\frac{1.962}{2}(2.353)\right)=(-1.937, 2.687)$

Non Normal Case (Large Samples)

For (disc. or cont.) large samples size ($n \geq 30$) selected from non-normal distns, we can find (with help of C.L.T) an approximate c.I for μ because most distns has limiting normal distn as $n \to \infty$.

Let $X_1, X_2, --, X_n$ be a r.s of size n ($n \geq 30$) from any distn that has mean μ, variance σ^2, and m.g.f $M(t)$ exist for $-h<t<h$ and suppose we require a c.I for μ with prob. $1-\alpha$.

There are two cases

Case(i) when σ^2 is known and $n \geq 30$ (n large)

According to C.L.T, we have the r.v $Z=\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \overset{app}{\sim} N(0,1)$

so, we can find from $N(0,1)$ distn table two no.s, say $\pm z_{1-\frac{\alpha}{2}}$ such that $Pr\left(-z_{1-\frac{\alpha}{2}}<Z<z_{1-\frac{\alpha}{2}}\right)=1-\alpha$.

Now

$-z_{1-\frac{\alpha}{2}}<Z<z_{1-\frac{\alpha}{2}} \equiv z_{1-\frac{\alpha}{2}}<\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}<z_{1-\frac{\alpha}{2}} \equiv \bar{X}-\frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}<\mu<\bar{X}+\frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$

$\therefore$ As given in the case (i) of normal case, the approximate $100(1-\alpha)\%$ c.I for μ is $\left(\bar{X}-\frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X}+\frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

Example (7) A simple of size 36 from a distn that has mean μ, variance $\sigma^2=144$, and its m.g.f exist. This sample yeilds $\bar{x}=67.53$. Find the approximate 95% c.I for μ.

Solution $n=36$ (large), $\sigma=12$, $\bar{x}=67.53$, so we approximate to normal

$1-\alpha=0.95 \Rightarrow \alpha=0.05 \Rightarrow \frac{\alpha}{2}=0.025 \Rightarrow 1-\frac{\alpha}{2}=0.975$

From $N(0,1)$ distn table, we have $z_{1-\frac{\alpha}{2}} \equiv z_{0.975}=1.96$

$\therefore$ The approximate 95% C.I for μ is $\left(\bar{x}-\frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{x}+\frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

$= \left(67.53-\frac{12}{6}(1.96), 67.53+\frac{12}{6}(1.96)\right)$

$= (63.61, 71.45)$

Case (ii) When σ^2 is unknown and $n \geq 30$ (n large)

Again, according to C.L.T, the r.v $U = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \overset{app.}{\sim} N(0,1)$

Since $S^2 \xrightarrow[Sto]{Conv.} \sigma^2 \Rightarrow \frac{S^2}{\sigma^2} \xrightarrow[Sto.]{Conv.} 1 \Rightarrow V = \frac{S}{\sigma} \xrightarrow[Sto.]{Conv.} \sqrt{1} = 1$

Then the r.v $Z = \frac{U}{V}$ has limiting distn as that of U.

That is

$$Z = \frac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{\frac{S}{\sigma}} = \frac{\sqrt{n}(\bar{X}-\mu)}{S} \overset{app.}{\sim} N(0,1)$$

So, we can find from $N(0,1)$ distn table two no.^{s}, say $\pm z_{1-\frac{\alpha}{2}}$ suchthat

$Pr\left(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$

$\therefore$ As given in the case (i) of normal case, the approximate $100(1-\alpha)\%$ C.I for μ is $\left(\bar{X}-\frac{S}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X}+\frac{S}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

<u>Example (8)</u> A sample of size 100 from a distn that has mean μ, variance σ^2 and its m.g.f exist, this sample observe $\bar{x}=16$ and $s^2=5.76$. Determine a 90% C.I for μ.

<u>Solution</u> We have $n=100$ (large), σ^2 unknown, $\bar{x}=16$, $s^2=5.76 \Rightarrow s=2.4$

So, we approximate to normal.

$1-\alpha = 0.9 \Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05 \Rightarrow 1-\frac{\alpha}{2} = 0.95$

From $N(0,1)$ distn table, we have $z_{1-\frac{\alpha}{2}} \equiv z_{0.95} = 1.645$

$\therefore$ The approximate 90% C.I for μ is $\left(\bar{x}-\frac{s}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{x}+\frac{s}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)$

$= \left(16-\frac{2.4}{10}(1.645), 16+\frac{2.4}{10}(1.645)\right)$

$= (15.605, 16.395)$

Non-Normal Case (Small samples)

For small samples size ($n<30$) selected from non-normal distns, we cannot expect our degree of confidence to be accurate. However, with assumption that the sample is approximately normal, we can derive an approximate $100(1-\alpha)\%$ C.I for μ when σ^2 is unknown.

The derivation exactly as given in case (ii) of normal case. That is, the approximate $100(1-\alpha)\%$ C.I for μ when σ^2 is unknown is

$$\left(\bar{X}-\frac{S}{\sqrt{n}}t_{1-\frac{\alpha}{2}}\ ,\ \bar{X}+\frac{S}{\sqrt{n}}t_{1-\frac{\alpha}{2}}\right)$$

Example (9) The length of 10 basket ball players in cm are:- 182, 199, 201, 189, 192, 190, 203, 178, 197, 187. Find a 95% C.I for μ. Assume approximate normal distn for the length.

Solution we have $n=10$ (n small), σ^2 unknown, the assumption for the length is approximately normal.

$$\bar{x}=\frac{1}{n}\sum x_i=\frac{1}{10}(182+199+\cdots+187)=191.4$$

$$s^2=\frac{1}{n-1}\left[\sum x_i^2-n\bar{x}^2\right]=\frac{1}{9}\left[(182)^2+(199)^2+\cdots+(187)^2-10(191.4)^2\right]$$

$$=67.733 \Rightarrow s=8.23$$

$$1-\alpha=0.95 \Rightarrow \alpha=0.05 \Rightarrow \frac{\alpha}{2}=0.025 \Rightarrow 1-\frac{\alpha}{2}=0.975$$

From t-distn table with $n-1=10-1=9$ dof, we have $t_{1-\frac{\alpha}{2}}=t_{0.975}=2.262$

So, the approximate 95% C.I for μ is $\left(\bar{x}-\frac{s}{\sqrt{n}}t_{1-\frac{\alpha}{2}}\ ,\ \bar{x}+\frac{s}{\sqrt{n}}t_{1-\frac{\alpha}{2}}\right)$

$$=\left(191.8-\frac{8.23}{\sqrt{10}}(2.262)\ ,\ 191.8+\frac{8.23}{\sqrt{10}}(2.262)\right)$$

$$=(185.9\ ,\ 203.6)$$

Summary For C.I of μ

Normal	Non-Normal approximated to Normal
$\bar{X} \pm \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$, if σ^2 is known	$\bar{X} \pm \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$, if σ^2 is known and $n \geq 30$
$\bar{X} \pm \frac{S}{\sqrt{n}} t_{1-\frac{\alpha}{2}}$, if σ^2 is unknown	$\bar{X} + \frac{S}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$, if σ^2 is unknown and $n \geq 30$
$\bar{X} \pm \frac{S}{\sqrt{n}} t_{1-\frac{\alpha}{2}}$, if σ^2 is unknown and $n<30$ with the assumption that the sample is approximately normal	

Confidence Intervals for Difference of Means

Normal case

Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ be two independent random samples of sizes n and m from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively. Suppose we require C.I for $\mu_1 - \mu_2$ with prob. $1-\alpha$.

The are two cases :-

Case(i) when σ_1^2 and σ_2^2 are knowns

For normal case we know that $\bar{X} \sim N(\mu_1, \frac{\sigma_1^2}{n})$ and $\bar{Y} \sim N(\mu_2, \frac{\sigma_2^2}{m})$

$\Rightarrow \bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m})$. Then the r.v

$$Z = \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \sim N(0,1)$$

Hence, we can find from $N(0,1)$ distn table two no.s, say $\pm Z_{1-\frac{\alpha}{2}}$ such that $Pr(-Z_{1-\frac{\alpha}{2}} < Z < Z_{1-\frac{\alpha}{2}}) = 1-\alpha$.

Now

$$-Z_{1-\frac{\alpha}{2}} < Z < Z_{1-\frac{\alpha}{2}} \equiv -Z_{1-\frac{\alpha}{2}} < \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} < Z_{1-\frac{\alpha}{2}}$$

$$\equiv (\bar{X}-\bar{Y}) - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} < \mu_1 - \mu_2 < (\bar{X}-\bar{Y}) + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}.$$

$\therefore$ The $100(1-\alpha)\%$ C.I for $\mu_1 - \mu_2$ is

$$\left((\bar{X}-\bar{Y}) - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} \,,\, (\bar{X}-\bar{Y}) + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} \right)$$

Example (10) Two independent samples of sizes 100 and 72 from $N(\mu_1, 25)$ and $N(\mu_2, 36)$ respectively. These sample yields $\bar{x} = 80$ and $\bar{y} = 75$. Find a 90% C.I for $\mu_1 - \mu_2$

Solution we have $n = 100$, $m = 72$, $\sigma_1^2 = 25$, $\sigma_2^2 = 36$, $\bar{x} = 80$, $\bar{y} = 75$

$1-\alpha = 0.9 \Rightarrow 1 - \frac{\alpha}{2} = 0.95$. From $N(0,1)$ distn table, we have $Z_{1-\frac{\alpha}{2}} = Z_{0.95} = 1.645$. The 90% C.I for $\mu_1 - \mu_2$ is

$$\left((\bar{x}-\bar{y}) - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} \,,\, (\bar{x}-\bar{y}) + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} \right)$$

$$\equiv\left((80-75)-(1.645)\sqrt{\frac{25}{100}+\frac{36}{72}}\ ,\ (80-75)+(1.645)\sqrt{\frac{25}{100}+\frac{36}{72}}\right)$$

$$\equiv\left(5-(1.645)\frac{\sqrt{3}}{2}\ ,\ 5+(1.645)\frac{\sqrt{3}}{2}\right)\equiv(3.575\ ,\ 6.425)$$

Case (ii) When σ_1^2 and σ_2^2 are unknown but $\sigma_1^2=\sigma_2^2=\sigma^2$

For normal case, we know the following

(1) $\bar{X}\sim N(\mu_1,\frac{\sigma^2}{n})$ and $\bar{Y}\sim N(\mu_2,\frac{\sigma^2}{m})$

(2) $\frac{(n-1)S_1^2}{\sigma^2}\sim\chi^2(n-1)$ and $\frac{(m-1)S_2^2}{\sigma^2}\sim\chi^2(m-1)$

(3) $\bar{X}$ and S_1^2 are sto indep and $\bar{Y}$ and S_2^2 are sto. indep

Now

$$\bar{X}-\bar{Y}\sim N\left(\mu_1-\mu_2,\frac{\sigma^2}{n}+\frac{\sigma^2}{m}\right)\Rightarrow\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}\sim N(0,1)$$

Also $\frac{(n-1)S_1^2}{\sigma^2}+\frac{(m-1)S_2^2}{\sigma^2}\sim\chi^2(n+m-2)$. Then the r-v

$$T=\frac{\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}}}{\sqrt{\frac{\frac{(n-1)S_1^2+(m-1)S_2^2}{\sigma^2}}{n+m-2}}}=\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\cdot\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}}\sim t(n+m-2)$$

Hence from t-distn table with $(n+m-2)$ dof, we can find two no's $\pm t_{1-\frac{\alpha}{2}}$ suchthat $Pr(-t_{1-\frac{\alpha}{2}}<T<t_{1-\frac{\alpha}{2}})=1-\alpha$

Now

$$-t_{1-\frac{\alpha}{2}}<T<t_{1-\frac{\alpha}{2}}\equiv-t_{1-\frac{\alpha}{2}}<\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\cdot\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}}<t_{1-\frac{\alpha}{2}}$$

$$\equiv(\bar{X}-\bar{Y})-t_{1-\frac{\alpha}{2}}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}<\mu_1-\mu_2<(\bar{X}-\bar{Y})+t_{1-\frac{\alpha}{2}}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}$$

$\therefore$ The $100(1-\alpha)\%$ C.I for $\mu_1-\mu_2$ is

$$\left((\bar{X}-\bar{Y})-t_{1-\frac{\alpha}{2}}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\cdot\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}\ ,\ (\bar{X}-\bar{Y})+t_{1-\frac{\alpha}{2}}\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\cdot\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}\right)$$

As given in case (i) of normal case, the approximate $100(1-\alpha)\%$ C.I for $\mu_1-\mu_2$ is

$$\left((\bar{X}-\bar{Y}) - z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}} \, , \, (\bar{X}-\bar{Y}) + z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}} \right)$$

Example (12) A r.s of size $n=100$ taken from a distn whose standard deviation $\sigma_1=5$ yield $\bar{x}=80$. A second r.s of size $m=75$ taken from a different distn whose standard deviation $\sigma_2=3$ yield $\bar{y}=72$. Find an approximate 90% C.I for $\mu_1-\mu_2$

Solution

$n=100$, $m=75$ (n and m are large), $\sigma_1^2=25$, $\sigma_2^2=9$, $\bar{x}=80$, $\bar{y}=72$
so, we approximate to normal

$1-\alpha=0.9 \Rightarrow 1-\frac{\alpha}{2}=0.95$. From $N(0,1)$ table, we have $z_{1-\frac{\alpha}{2}}=z_{0.95}=1.645$

$\therefore$ The approximate 90% C.I for $\mu_1-\mu_2$ is.

$$\left((\bar{x}-\bar{y}) - z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}} \, , \, (\bar{x}-\bar{y}) + z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}} \right)$$

$$\equiv \left((80-72)-(1.645)\sqrt{\frac{25}{100}+\frac{9}{75}} \, , \, (80-72)+(1.645)\sqrt{\frac{25}{100}+\frac{9}{75}} \right)$$

$$\equiv \left(8-(1.645)\sqrt{0.37} \, , \, 8+(1.645)\sqrt{0.37} \right) \equiv (6.999 \, , \, 9.001)$$

Case (ii) when σ_1^2 and σ_2^2 are unknown and $n \geq 30$, $m \geq 30$

According to C.L.T, the r.v

$$U = \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}} \overset{app}{\sim} N(0,1)$$

since $S_1^2 \xrightarrow[sto]{conv} \sigma_1^2 \Rightarrow \frac{S_1^2}{n} \xrightarrow[sto]{conv} \frac{\sigma_1^2}{n}$ and $S_2^2 \xrightarrow[sto]{conv} \sigma_2^2 \Rightarrow \frac{S_2^2}{m} \xrightarrow[sto]{conv} \frac{\sigma_2^2}{m}$

$$\Rightarrow \frac{S_1^2}{n}+\frac{S_2^2}{m} \xrightarrow[sto.]{conv.} \frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m} \Rightarrow \frac{\frac{S_1^2}{n}+\frac{S_2^2}{m}}{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}} \xrightarrow[sto]{conv.} 1$$

and the r.v

$$V = \sqrt{\frac{\frac{S_1^2}{n}+\frac{S_2^2}{m}}{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}} \xrightarrow[sto.]{conv.} \sqrt{1}=1$$

$\therefore$ The r.v $Z=\frac{U}{V}$ has limiting as that of U. That is

Example (11) A r.s of size 10 from $N(\mu_1, \sigma^2)$ yield $\bar{x} = 4.2$ and $S_1^2 = 49$. Another r.s (indep. of the 1st sample) of size 7 from $N(\mu_2, \sigma^2)$ yield $\bar{y} = 3.4$ and $S_2^2 = 32$. Find a 90% c.I for $\mu_1 - \mu_2$.

Solution

$n = 10,\ m = 7,\ \sigma_1^2 = \sigma_2^2 = \sigma^2,\ \bar{x} = 4.2,\ S_1^2 = 49,\ \bar{y} = 3.4,\ S_2^2 = 32.$

$1 - \alpha = 0.9 \Rightarrow 1 - \frac{\alpha}{2} = 0.95$. From t-distn table with $n+m-2 = 15$ d.f, we have $t_{1-\frac{\alpha}{2}} \equiv t_{0.95} = 1.753$

$\therefore$ The 90% c.I for $\mu_1 - \mu_2$ is

$$\left((\bar{x}-\bar{y}) - t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right) \cdot \frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2}},\ (\bar{x}-\bar{y}) + t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right) \cdot \frac{(n-1)S_1^2 + (m-1)S_2^2}{n+m-2}} \right)$$

$$\equiv \left((4.2-3.4) - (1.753) \sqrt{\left(\frac{1}{10}+\frac{1}{7}\right) \cdot \frac{(9)(49)+(6)(32)}{15}},\ (4.2-3.4) + (1.753) \sqrt{\left(\frac{1}{10}+\frac{1}{7}\right) \cdot \frac{(9)(49)+(6)(32)}{15}} \right)$$

$$\equiv \left(0.8 - (1.753) \sqrt{\frac{17}{70}(42.2)},\ 0.8 - (1.753) \sqrt{\frac{17}{70}(42.2)} \right) \equiv (-4.812, 6.412)$$

Non-Normal Case (Large samples), $n \geq 30$, $m \geq 30$

If two independent large samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ of sizes n and m are selected from non-normal distns with means μ_1 and μ_2, variances σ_1^2 and σ_2^2 and the m.g.f's are exist, we can find with help of C.L.T an approximate c.I for $\mu_1 - \mu_2$ with prob. $1 - \alpha$.

There are two cases :-

Case (i) when σ_1^2 and σ_2^2 are known and $n \geq 30$, $m \geq 30$.

According to C.L.T, the r.v's

$$\frac{\sqrt{n}(\bar{X}-\mu_1)}{\sigma_1} \overset{app}{\sim} N(0,1) \Rightarrow \bar{X} \overset{app.}{\sim} N\left(\mu_1, \frac{\sigma_1^2}{n}\right)$$

and

$$\frac{\sqrt{m}(\bar{Y}-\mu_2)}{\sigma_2} \overset{app}{\sim} N(0,1) \Rightarrow \bar{Y} \overset{app.}{\sim} N\left(\mu_2, \frac{\sigma_2^2}{m}\right)$$

$$\Rightarrow \bar{X} - \bar{Y} \overset{app.}{\sim} N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right).$$ Then the r.v

$$Z = \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \overset{app}{\sim} N(0,1)$$

$$Z = \frac{\dfrac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\left(\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}\right)^{1/2}}}{\left(\dfrac{\frac{S_1^2}{n}+\frac{S_2^2}{m}}{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}\right)^{1/2}} = \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}}} \overset{app}{\sim} N(0,1)$$

As given in case (i) of normal case, the approximate $100(1-\alpha)\%$ c.I for $\mu_1-\mu_2$ is

$$\left((\bar{X}-\bar{Y}) - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}} \;,\; (\bar{X}-\bar{Y}) + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}} \right)$$

Example (13) A maths test was given to 75 boys and 50 girls. The boys made an average grade of 82 with standard deviation of 8, while the girls made an average grade of 76 with standard deviation of 6. Find a 96% c.I for $\mu_1-\mu_2$ where μ_1 is the mean score of all boys and μ_2 is the mean score of all girls.

Solution

$n=75$, $m=50$ (n and m are large), σ_1^2 and σ_2^2 are unknown, $\bar{x}=82$, $\bar{y}=76$, $S_1=8$, $S_2=6$, we approximate to normal

$1-\alpha=0.96 \Rightarrow 1-\frac{\alpha}{2}=0.98$. From $N(0,1)$ table, we have

$Z_{1-\frac{\alpha}{2}} = Z_{0.98} = 2.05$

$\therefore$ The approximate 96% c.I for $\mu_1-\mu_2$ is

$$\left((\bar{x}-\bar{y}) - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}} \;,\; (\bar{x}-\bar{y}) + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}} \right)$$

$$\equiv \left((82-76)-(2.05)\sqrt{\frac{64}{75}+\frac{36}{50}} \;,\; (82-76)+(2.05)\sqrt{\frac{64}{75}+\frac{36}{50}} \right)$$

$$\equiv \left(6-(2.05)(2.57)^{1/2}, \; 6+(2.05)(2.57)^{1/2} \right) \equiv (2.714, \; 9.286)$$

Non-Normal (Small Samples)

For small samples sizes ($n<30$, $m<30$) selected from non-normal distns with assumption that the samples are approximately normal, the derivation of an approximate $100(1-\alpha)\%$ c.I for $\mu_1-\mu_2$ is exactly

as given in case (ii) for normal case when the unknowns $\sigma_1^2 = \sigma_2^2 = \sigma^2$.
That is, the approximate $100(1-\alpha)\%$ C.I for $\mu_1 - \mu_2$ is

$$\left((\bar{X}-\bar{Y}) - t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}\right)}, (\bar{X}-\bar{Y}) + t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}\right)} \right)$$

Example (14) Records of the past 15 years have shown that the average rainfall in a certain region for the month May is 4.93 cm with standard deviation of 1.14 cm. A second region has had an average rainfall in May 2.64 cm with standard deviation of 0.66 cm during the past 10 years. Find a 95% C.I for the different of the true average rainfalls in these two regions. Assume that the observations come from an approximate two independent normal distn with equal variances.

Solution

$n=15, m=10$ $(n<30, m<30)$, $\sigma_1^2 = \sigma_2^2 = \sigma^2$ unknown, $\bar{x} = 4.93, \bar{y} = 2.64$
$s_1 = 1.14$, $s_2 = 0.66$, the assumption is approximately normal.
$1-\alpha = 0.95 \Rightarrow 1-\frac{\alpha}{2} = 0.975$. From t-distn table with $n+m-2 = 15+10-2 = 23$ d.f, we have $t_{1-\frac{\alpha}{2}} = t_{0.975} = 2.069$

$\therefore$ The approximate 95% C.I for $\mu_1 - \mu_2$ is

$$\left((\bar{x}-\bar{y}) - t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{(n-1)s_1^2+(m-1)s_2^2}{n+m-2}\right)}, (\bar{x}-\bar{y}) + t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{(n-1)s_1^2+(m-1)s_2^2}{n+m-2}\right)} \right)$$

$$\equiv (4.93-2.64) \mp (2.069) \sqrt{\left(\frac{1}{15}+\frac{1}{10}\right)\left(\frac{(14)(1.14)^2 + 9(0.66)^2}{23}\right)}$$

$$\equiv 2.29 \pm (2.069)(0.4) \equiv (1.89, 2.69)$$

Summary of C.I For $\mu_1-\mu_2$

Normal

$$(\bar{X}-\bar{Y}) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}$$, For known σ_1^2 and σ_2^2

$$(\bar{X}-\bar{Y}) + t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}\right)}$$, For unknow σ_1^2 and σ_2^2 with $\sigma_1^2=\sigma_2^2=\sigma^2$

Non-Normal Approximated To Normal (Large Samples

$$(\bar{X}-\bar{Y}) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}$$, For known σ_1^2, σ_2^2 and $n \geq 30, m \geq 30$

$$(\bar{X}-\bar{Y}) \pm Z_{1-\frac{\alpha}{2}} \sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}}$$, For unknown σ_1^2, σ_2^2 and $n \geq 30, m \geq 30$

Non-Normal (Small Samples)

$$(\bar{X}-\bar{Y}) \pm t_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}\right)}$$

For unknown σ_1^2, σ_2^2 and $n<30, m<30$ with $\sigma_1^2=\sigma_2^2=\sigma^2$ and the assumption that the samples are approximately normal

Confidence Interval For the Variance σ^2

Normal Case

Let $X_1, X_2, \dots, X_n$ be a r.s of size n from $N(\mu, \sigma^2)$. Suppose, we wish to find C.I for the variance σ^2 with prob $1-\alpha$.

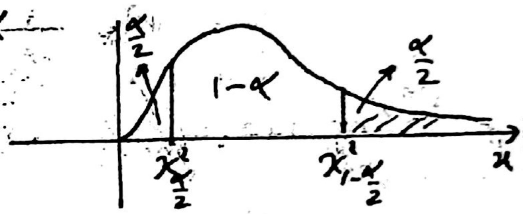
There are two cases :-

Case (i). When μ is known

For $i = 1, 2, \dots, n$, we have $X_i \sim N(\mu, \sigma^2) \Rightarrow \frac{X_i - \mu}{\sigma} \sim N(0,1)$

$$\Rightarrow \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)} \Rightarrow Y = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

Hence, we can find from χ^2-distn table with n dof two numbers, say, $\chi^2_{\frac{\alpha}{2}}$ and $\chi^2_{1-\frac{\alpha}{2}}$ suchthat $P_r\left(\chi^2_{\frac{\alpha}{2}} < Y < \chi^2_{1-\frac{\alpha}{2}}\right) = 1-\alpha$



Now $\chi^2_{\frac{\alpha}{2}} < Y < \chi^2_{1-\frac{\alpha}{2}} \equiv \chi^2_{\frac{\alpha}{2}} < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} < \chi^2_{1-\frac{\alpha}{2}}$

$$\equiv \frac{1}{\chi^2_{1-\frac{\alpha}{2}}} < \frac{\sigma^2}{\sum_{i=1}^{n}(X_i-\mu)^2} < \frac{1}{\chi^2_{\frac{\alpha}{2}}}$$

$$\equiv \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{\frac{\alpha}{2}}}$$

$\therefore$ The $100(1-\alpha)\%$ C.I for σ^2 is $\left(\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{1-\frac{\alpha}{2}}}, \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{\frac{\alpha}{2}}}\right)$

EX(15)

A r.s of size 10 from $N(0, \sigma^2)$ yield $\sum_{i=1}^{10} x_i^2 = 106.6$. Find a 95% C.I for σ^2

Solution : $n = 10$, $\mu = 0$ (known), $\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{10}(x_i-0)^2 = \sum_{i=1}^{10} x_i^2 = 106.6$

$1-\alpha = 0.95 \Rightarrow 1-\frac{\alpha}{2} = 0.975 \Rightarrow \frac{\alpha}{2} = 0.025$. From χ^2-distn table with $n=10$ dof, we have

$$\chi^2_{\frac{\alpha}{2}} = \chi^2_{0.025} = 3.25 \text{ and } \chi^2_{1-\frac{\alpha}{2}} = \chi^2_{0.975} = 20.5$$

$\therefore$ The 95% C.I for σ^2 is $\left(\dfrac{\sum_{i=1}^{n}(x_i-\mu)^2}{\chi^2_{1-\frac{\alpha}{2}}}, \dfrac{\sum_{i=1}^{n}(x_i-\mu)^2}{\chi^2_{\frac{\alpha}{2}}} \right)$.

$$\equiv \left(\frac{106.6}{20.5}, \frac{106}{3.25} \right) \equiv (5.2, 32.5)$$

Case (ii) when μ is unknown

For normal case, we know that the r.v $Y = \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

So, we can find from χ^2-distn with n-1 dof two numbers, say, $\chi^2_{\frac{\alpha}{2}}$ and $\chi^2_{1-\frac{\alpha}{2}}$ suchthat $Pr(\chi^2_{\frac{\alpha}{2}} < Y < \chi^2_{1-\frac{\alpha}{2}}) = 1-\alpha$

Now $\chi^2_{\frac{\alpha}{2}} < Y < \chi^2_{1-\frac{\alpha}{2}} \equiv \chi^2_{\frac{\alpha}{2}} < Y < \chi^2_{1-\frac{\alpha}{2}} \equiv \chi^2_{\frac{\alpha}{2}} < \dfrac{(n-1)S^2}{\sigma^2} < \chi^2_{1-\frac{\alpha}{2}}$

$$\equiv \frac{1}{\chi^2_{1-\frac{\alpha}{2}}} < \frac{\sigma^2}{(n-1)S^2} < \frac{1}{\chi^2_{\frac{\alpha}{2}}} \equiv \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}}$$

$\therefore$ The $100(1-\alpha)\%$ C.I for σ^2 is $\left(\dfrac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}}, \dfrac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}} \right)$

Ex(16) The volumes of 10 cans distributed by a certain company are:
46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, 46.0.
Find a 95% C.I for the variance of all such cans if the volume distn is $N(\mu, \sigma^2)$.

Solution : $n = 10$, μ is unknown, $\bar{x} = \frac{1}{n}\sum_{i=1}^{10} x_i = \frac{1}{10}(46.4 + 46.1 + \dots + 46.0) = 46.12$

$\sum_{i=1}^{n} x_i^2 = (46.4)^2 + (46.1)^2 + \dots + (46.0)^2 = 21273.13$

$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right] = \frac{1}{9}\left[21273.13 - (10)(46.12)^2\right] = 0.286$

$(n-1)S^2 = 2.576$

$1-\alpha = 0.95 \Rightarrow 1-\frac{\alpha}{2} = 0.975 \Rightarrow \frac{\alpha}{2} = 0.025$. From χ^2 distn table with $n-1 = 10-1 = 9$ dof, we have $\chi^2_{\frac{\alpha}{2}} = \chi^2_{0.025} = 2.7$ and $\chi^2_{1-\frac{\alpha}{2}} = \chi^2_{0.975} = 19.023$

$\therefore$ The 95% C.I for σ^2 is $\left(\dfrac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}}, \dfrac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}} \right) \equiv \left(\dfrac{2.576}{19.023}, \dfrac{2.576}{2.7} \right)$

$$\equiv (0.135, 0.954)$$

Confidence Interval For the Ratio of two Variances

Normal Case

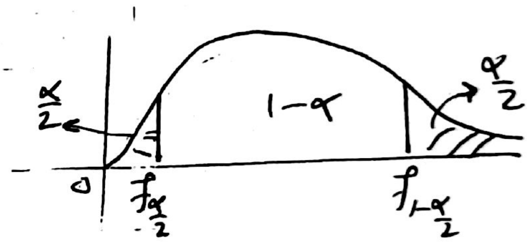
Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ be two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively. Suppose we wish to find a C.I for the ratio $\frac{\sigma_2^2}{\sigma_1^2}$ with prob. $1-\alpha$

For Normal case, we have $\frac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2_{(n-1)}$ and $\frac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2_{(m-1)}$
where S_1^2 and S_2^2 are the samples variance.

Then the r.v

$$F = \frac{\frac{(n-1)S_1^2}{\sigma_1^2} \Big/ (n-1)}{\frac{(m-1)S_2^2}{\sigma_2^2} \Big/ (m-1)} = \frac{\sigma_2^2}{\sigma_1^2}\,\frac{S_1^2}{S_2^2} \sim F(n-1, m-1)$$

So, we can find from F-distn table with $(n-1, m-1)$ dof ~~two~~ number, say,
$f_{\frac{\alpha}{2}}$, $f_{1-\frac{\alpha}{2}}$ suchthat $Pr\left(f_{\frac{\alpha}{2}} < F < f_{1-\frac{\alpha}{2}}\right) = 1-\alpha$



Now $f_{\frac{\alpha}{2}} < F < f_{1-\frac{\alpha}{2}} \equiv f_{\frac{\alpha}{2}} < \frac{\sigma_2^2}{\sigma_1^2}\frac{S_1^2}{S_2^2} < f_{1-\frac{\alpha}{2}}$

$$\equiv \frac{S_2^2}{S_1^2} f_{\frac{\alpha}{2}} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{S_2^2}{S_1^2} f_{1-\frac{\alpha}{2}}$$

$\therefore$ The $100(1-\alpha)\%$ C.I for $\frac{\sigma_2^2}{\sigma_1^2}$ is $\left(\frac{S_2^2}{S_1^2} f_{\frac{\alpha}{2}}, \frac{S_2^2}{S_1^2} f_{1-\frac{\alpha}{2}}\right)$

Remark $f_{\frac{\alpha}{2}}^{(n,m)} = \frac{1}{f_{1-\frac{\alpha}{2}}^{(m,n)}}$

EX(17) Two independent samples of sizes $n=16$, $m=10$ from two independent $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ distns yields $S_1^2 = 4.14$, $S_2^2 = 7.25$.
Find a 90% C.I for $\frac{\sigma_2^2}{\sigma_1^2}$

Solution

$n=16$, $m=10$, $S_1^2 = 4.14$, $S_2^2 = 7.25$

$1-\alpha = 0.90 \Rightarrow 1-\frac{\alpha}{2} = 0.95 \Rightarrow \frac{\alpha}{2} = 0.05$

From F-distn table with $(n-1, m-1) = (15, 9)$ dof, we have

$f_{1-\frac{\alpha}{2}}^{(15,9)} = f_{0.95}^{(15,9)} = 3.01$, $f_{\frac{\alpha}{2}}^{(15,9)} = f_{0.05}^{(15,9)} = \frac{1}{f_{0.95}^{(9,15)}} = \frac{1}{2.59} = 0.386$

$\therefore$ The 90% C.I for $\frac{\sigma_2^2}{\sigma_1^2}$ is $\left(\frac{S_2^2}{S_1^2} f_{\frac{\alpha}{2}} , \frac{S_2^2}{S_1^2} f_{1-\frac{\alpha}{2}}\right)$

$\equiv \left(\frac{7.26}{4.14}(0.386) , \frac{7.26}{4.14}(3.01)\right) \equiv (0.677 , 5.28)$