
بسم الله الرحمن الرحيم
In The Name Of Allah The Most Beneficent, The Most Merciful

المحاضرة
الحادية عشر

The Cauchy – Riemann Equations

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Cauchy-Riemann Equations

- Theorem 1 :- Necessary conditions for a function f(z) to be analytic. If $f(z) = u(x, y) + i\, v(x, y)$ is analytic at z_0, then u and v satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad i.e.\, u_x = v_y$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad i.e.\, v_x = -u_y$$

at every point in some neighbourhood of point z_0 provided u_x, u_y, v_x & v_y exists.

Cauchy-Riemann Equations

- Theorem 2 :- Sufficient condition for f(z) to be analytic, If

1) $f(z) = u(x, y) + i\, v(x, y)$ is defined at every point in some neighbourhood of point z_0.
2) $u\ and\ v$ satisfy CR equations $u_x = v_y,\ v_x = -u_y$ at every point in some neighbourhood of point z_0.
3) u, v, u_x, u_y, v_x, v_y are continuous at every point in some neighbourhood of point z_0.

Then the function f(z) is analytic at z_0 & $f'(z) = u_x + iv_x$

Applying the Cauchy – Riemann Conditions

◊ **Example 1:**

$$f(z) = z = (x+iy) = \underbrace{x}_{u(x,y)} + i \underbrace{y}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} \quad ✓$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \quad ✓$$

$\Rightarrow$ **C.R. conditions hold everywhere (for z finite)**

$\Rightarrow$ z **is analytic everywhere**

◊ **Example 2 :**

$$f(z) = z^* = (x+iy)^* = \underbrace{x}_{u(x,y)} + i \overbrace{(-y)}^{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \quad ✗$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \quad ✓$$

$\Rightarrow$ **C.R. conditions hold *nowhere***

$\Rightarrow$ z^* **is analytic nowhere**

Applying the Cauchy – Riemann Conditions (cont.)

◊ **Example 3 :**

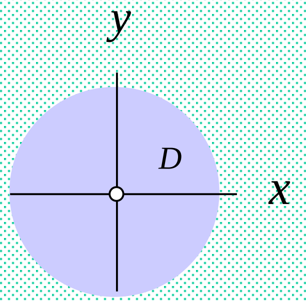
$$f(z)=\frac{1}{z}=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$$

$$=\underbrace{\frac{x}{x^2+y^2}}_{u(x,y)}+i\underbrace{\left(\frac{-y}{x^2+y^2}\right)}_{v(x,y)}$$

$$\frac{\partial u}{\partial x}=\frac{\cancel{x^2}+y^2-\cancel{2}x^2}{\left(x^2+y^2\right)^2}\overset{?}{=}\frac{\partial v}{\partial y}=\frac{-x^2-\cancel{y^2}+\cancel{2}y^2}{\left(x^2+y^2\right)^2} \quad ✓$$

$$\frac{\partial u}{\partial y}=\frac{-2xy}{\left(x^2+y^2\right)^2}=-\frac{\partial v}{\partial x} \quad ✓$$

$\Rightarrow$ **C.R. conditions hold everywhere except** $x=y=0$ $(z=0)$.



$\frac{1}{z}$ is analytic except at $z=0$

$\Rightarrow \quad D: |z|>0$

$f(z)$ **is analytic everywhere except at** $z=0$. **The point** $z=0$ **is called a "singularity."**

A singularity is a point where the function is <u>not</u> analytic.

Check each function for analyticity by using the Cauchy-Riemann equations.

(a) $f(z) = |z|^2$

(b) $g(z) = e^{z^2}$

(c) $h(z) = \operatorname{Im}(z^2)$

(a) $f(z) = |z|^2 = x^2 + y^2$.

Hence, $u = x^2 + y^2$ and $v = 0$. Now $u_x = 2x \neq 0 = v_y$.

Cauchy-Riemann equations fail. The function is **not analytic**.

Cauchy-Riemann Equations

- Polar form :- variables $r\ and\ \theta$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ \ i.e.\, u_r = \frac{1}{r}v_\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}\ \ i.e.\, \frac{-1}{r}u_\theta = v_r$$

$$f'(z) = (u_r + i\ v_r)e^{-i\theta}$$

END