الدوال المثلثية

TRIGONOMETRIC FUNCTIONS

د.حكمت شريف مصطفى

المحاضرة الرابعةعشر

Trigonometric and Hyperbolic Functions. Euler's Formula

$$e^{ix} = (\cos x + i \sin x) \qquad e^{-ix} = (\cos x - i \sin x)$$

By addition and subtraction we obtain

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Trigonometric and Hyperbolic Functions. Euler's Formula

- Substitute (z= x+iy) instead of x, we obtain

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

- functions in this formula are unrelated in real

DEFINITION 17.11 Trigonometric Sine and Cosine

For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad (2)$$

- Four additional trigonometric functions:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z}, \qquad (3)$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

- Since e^{iz} and e^{-iz} are entire functions, then $\sin z$ and $\cos z$ are entire functions. Besides, $\sin z = 0$ only for the real numbers $z = n\pi$ and $\cos z = 0$ only for the real numbers $z = (2n+1)\pi/2$. Thus $\tan z$ and $\sec z$ are analytic except $z = (2n+1)\pi/2$, and $\cot z$ and $\csc z$ are analytic except $z = n\pi$.

Derivatives

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

Similarly we have

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z \qquad (4)$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

Identities

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2 \sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z$$

Zeros

- If y is real, we have

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2} \qquad (5)$$

From Definition 11.17 and Euler's formula

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \sin x\left(\frac{e^y + e^{-y}}{2}\right) + i\cos x\left(\frac{e^y - e^{-y}}{2}\right)$$

Thus we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y \tag{6}$$

and

$$\cos z = \cos x \cosh y - i \sin x \sinh y \tag{7}$$

From (6) and (7) and $\cosh^2 y = 1 + \sinh^2 y$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \tag{8}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \tag{9}$$

Example 2

Solve cos $z = 10$.

Solution

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 10$$

$$e^{2iz} - 20e^{iz} + 1 = 0, e^{iz} = 10 \pm 3\sqrt{11}$$

$$iz = \log_e(10 \pm 3\sqrt{11}) + 2n\pi i$$

Since $\log_e(10 - 3\sqrt{11}) = -\log_e(10 + 3\sqrt{11})$

we have

$$z = 2n\pi \pm i\log_e(10 + 3\sqrt{11})$$

PROBLEM

Function Values. Find, in the form $u + iv$,

$\cos i, \quad \sin i$

Equations. Find all solutions.

$$\sin z = 100$$

Re tan z and Im tan z. Show that

$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$


Thank You