المحاضرة التاسعة

DERIVATIVES OF COMPLEX FUNCTIONS

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Derivative

Definition The **derivative** of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. Then f is said to be **differentiable** at z_0. If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and (4) takes the form

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

EXAMPLE **Differentiability. Derivative**

The function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

Some Exercises

Show that

1) $f(z)=\bar{z}$ is nowwhere differentiable.
2) $g(z)=z^n$ has derivative nz^{n-1}.
3) $h(z)=e^z$ has derivative e^z.
4) $l(z)=|z|^2$ is nowhere differentiable except $z=0$
5) Every real-valued function of complex variable is either non-differentiable or differentiable with derivative equal to 0.

solution

1) We study the limit for two different paths of the point, so if the first approach to the point is z0 along the line parallel to the real axis then $z = x + iy_0$, and therefore is

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{(x+iy_0)\to(x_0+iy_0)}\frac{f(x+iy_0)-f(x_0+iy_0)}{(x+iy_0)-(x_0+iy_0)}$$

$$= \lim_{(x+iy_0)\to(x_0+iy_0)}\frac{(x-iy_0)-(x_0-iy_0)}{(x-x_0)+i(y_0-y_0)}$$

$$= \lim_{(x+iy_0)\to(x_0+iy_0)}\frac{x-x_0}{x-x_0} = 1$$

as for if the approach to the point z0 along the line parallel to the imaginary axis y then $z = x_0 + iy$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{(x_0 + iy) - (x_0 + iy_0)}$$

$$= \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{(x_0 - iy) - (x_0 - iy_0)}{(x_0 - x_0) + i(y - y_0)}$$

$$= \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{-i(y - y_0)}{i(y - y_0)} = -1$$

From above, we find that

$f(z) = \overline{z}$ Is a non-differentiable.

2) $g(z)=z^n$ has derivative nz^{n-1}.

$$(z^n)' = \lim_{h\to 0} \frac{(z+h)^n - z^n}{h} = \lim_{h\to 0} \frac{z^n + nhz^{n-1} + h^2(\ldots) - z^n}{h}$$

$$= \lim_{h\to 0} \frac{nhz^{n-1} + h^2(\ldots)}{h} = \lim_{h\to 0} \left(nz^{n-1} + h(\ldots)\right) = nz^{n-1}$$

- Properties of Derivatives

$$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$$

$$(cf)'(z_0) = cf'(z_0) \text{ for any constant } c.$$

$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}, \text{ if } g(z_0) \neq 0.$$

$$\frac{d}{dz} f[g(z_0)] = f'[g(z_0)]g'(z_0) \text{ Chain Rule.}$$