المحاضرة الثالثة عشر

logarithmic Function

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Logarithm. General power. Principal value

- The natural logarithm of z = x+ iy is donated by ln z or log z

$$z = x + iy = r\,e^{i\theta}$$

$$ln\,z = ln\,(x + iy) = ln\,(r\,e^{i\theta})$$

$$ln\,z = ln\,r + i\theta$$

Logarithm. General power. Principal value

$$\ln z = \ln r + i\theta$$

- Where $r = |z| > 0, \theta = \arg z$
- the complex natural logarithm is infinitely many-valued. The value of ln z corresponding to the principle value Arg z is denoted by Ln z and it is called the principle value of ln z, given by

$$Ln\, z = \ln z + i\, Arg\, z$$

Logarithm. General power. Principal value

$$Ln\, z = \ln z + i\, Arg\, z$$

- Ln z is a single- valued since the other values of arg z differ by integer multiples of 2π. the other values of ln z given by:

$$ln\, z = \mathrm{Ln}\, z \pm 2n\pi$$

- All the values of ln z have the same real part but their imaginary parts differ by integer multiples of 2π .If z is a positive real then Arg z= 0 and Ln z becomes same as the real natural logarithm in calculus but if it is a negative number then Arg z $=\pi$ and $Ln\, z = \ln|z| +$

Example 2

Find the values of (a) ln (−2) (b) ln i, (c) ln (−1 − i).

Solution

$$(a)\quad \theta = \arg(-2) = \pi,\ \log_e |-2| = 0.6932$$

$$\ln(-2) = 0.6932 + i(\pi + 2n\pi)$$

$$(b)\quad \theta = \arg(i) = \frac{\pi}{2},\ \log_e 1 = 0$$

$$\ln(i) = i(\frac{\pi}{2} + 2n\pi)$$

$$(c)\quad \theta = \arg(-1-i) = \frac{5\pi}{4},\ \log_e |-1-i| = \log_e \sqrt{2} = 0.3466$$

$$\ln(-1-i) = 0.3466 + i(\frac{5\pi}{4} + 2n\pi)$$

Example 3

Find all values of z such that $e^z = \sqrt{3} + i$.

Solution

$$z = \ln(\sqrt{3} + i), |\sqrt{3} + i| = 2, \arg(\sqrt{3} + i) = \frac{\pi}{6}$$

$$z = \ln(\sqrt{3} + i) = \log_e 2 + i(\frac{\pi}{6} + 2n\pi)$$

$$= 0.6931 + i(\frac{\pi}{6} + 2n\pi)$$

Logarithm. General power. Principal value

- From $\ln z = \ln r + i\theta$ and $e^{\ln r} = r$ for positive real r we obtain

$$e^{\ln z} = z$$

- Since arg $(e^{z}) = y \pm 2n\pi$ is multivalued so:

$$\ln(e^{z}) = z \pm 2n\pi i$$

Logarithm. General power. Principal value

- The familiar relations of natural logarithm are still applicable for complex value:

$$a)\ \ln(z1z2) = \ln z1 + \ln z2 \qquad b)\ \ln\frac{z1}{z2} = \ln z1 - lnz2$$

Logarithm. General power. Principal value

- Analyticity of the Logarithm

Analyticity of the Logarithm

For every $n = 0, \pm 1, \pm 2, \cdots$ *formula* (3) *defines a function, which is analytic, except at* 0 *and on the negative real axis, and has the derivative*

$$(\ln z)' = \frac{1}{z} \qquad (z \text{ not } 0 \text{ or negative real}). \tag{6}$$

EXERCISES

1. Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

2. Show that, for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare Exercise 1.)

3. Verify expression (4), Sec. 31, for $\log(z_1/z_2)$ by
   (a) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 7);

EXERCISES

1. Show that

(*a*) $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$; (*b*) $\text{Log}(1 - i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

2. Verify that when $n = 0, \pm 1, \pm 2, \ldots,$

(*a*) $\log e = 1 + 2n\pi i$; (*b*) $\log i = \left(2n + \frac{1}{2}\right)\pi i$;

(*c*) $\log(-1 + \sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$.

3. Show that

(*a*) $\text{Log}(1 + i)^2 = 2\,\text{Log}(1 + i)$; (*b*) $\text{Log}(-1 + i)^2 \neq 2\,\text{Log}(-1 + i)$.

THANK YOU