المحاضرة الثانية عشر

CHAPTER 3

ELEMENTARY FUNCTIONS

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Elementary Complex Functions

In calculus, several derivative formulas have been established for the elementary functions of real variables, such as the exponential, trigonometric, logarithm, hyperbolic, and the inverse functions. In this chapter we shall extend the definitions of the elementary functions from real variables to complex variables and obtain derivative formulas for them. We begin with the construction of a suitable definition for the complex exponential function, which forms a basis for defining other elementary functions of complex variables.

EXPONENTIAL FUNCTION

Exponential Function

- The complex exponential function is one of the most important analytic functions

$$e^{z} = e^{x}\,(\cos y + i \sin y)$$

- If $z = 3 + 4i$ then

$$e^{z} = e^{3}\,(\cos 4 + i \sin 4)$$

Exponential Function

- For real z = x, imaginary part y = 0

$$e^{z} = e^{x}\,(\underbrace{\cancel{\cos}\, y}_{1} + i\,\underbrace{\cancel{\sin}\, y}_{0})$$

$$e^{z} = e^{x}$$

- e^{z} is analytic for all z

Complex exponential function

(a) e^z reduces to e^x when $\mathrm{Im}\, z = 0$.

(b) $e^z\, e^w = e^{z+w}$

(c) e^z is analytic.

(d) $\dfrac{d}{dz} e^z = e^z$.

Exponential Function

- The derivative of the exponential function is:

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x$$

$$= e^x \cos y + ie^x \sin y = e^z.$$

$$(e^z)' = e^z.$$

Properties of e^z

Theorem

Suppose z, w are complex numbers and n is a positive integer. Then

1. $e^z e^w = e^{z+w}$
2. $\dfrac{e^z}{e^w} = e^{z-w}$
3. $(e^z)^n = e^{nz}$
4. $|e^z| = e^x = e^{\mathrm{Re}\, z}$
5. *e^z is periodic with the imaginary period $2\pi i$*
6. *If k is an integer, then*
   6.1 *$e^z = 1$ if and only if $z = 2k\pi i$.*
   6.2 *$e^z = e^w$ if and only if $z = w + 2k\pi i$.*
7. *If $g(z)$ is analytic, then $\frac{d}{dz} e^{g(z)} = e^{g(z)} g'(z)$.*

Exponential Function

- General rule of the exponential functions that $e^a \times e^b = e^{(a+b)}$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)\,e^{x_2}(\cos y_2 + i\sin y_2).$$

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)] = e^{z_1+z_2}$$

Exponential Function

- Since $z = x + iy$

$$e^z = e^{(x+iy)} = e^x\, e^{iy}$$

- For pure imaginary complex number where $z = iy$

$$e^{iy} = \cos y + i \sin y.$$ Euler Formula

- The polar form of a complex number, $z = r(\cos\Theta + i \sin\Theta)$

  Can be written:

$$z = re^{i\theta}.$$

Exponential Function

- Substitution of 2π in

$$e^{iy} = cos\,(y) + i\,sin\,(y)$$

$$e^{2\pi i} = \underbrace{cos\,(2\pi)}_{1} + i\,\underbrace{sin\,(2\pi)}_{0}$$

$$e^{2\pi i} = 1$$

- Substitution of $y = \frac{\pi}{2}, \pi, \frac{-\pi}{2}\ and - \pi$ will yield

$$e^{\pi i/2} = i \qquad e^{\pi i} = -1 \qquad e^{-\pi i/2} = -i \qquad e^{-\pi i} = -1$$

Exponential Function

$$e^{iy} = \cos y + i \sin y.$$

$$\left|e^{iy}\right| = \left|\cos y + i \sin y\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

For pure imaginary exponent the exponential function has absolute value of 1

Q: What is the absolute value of exponential function if x doesn't equal to zero ?

Exponential Function

- Periodicity of e^x with period

$$e^{z+2\pi i} = e^z$$

$$e^{z+2\pi i} = e^z \times \underbrace{e^{2\pi i}}_{1}$$

Exponential Function

- Solve $e^z = 3 + 4i$:

$$|e^z| = \sqrt{3^2 + 4^2} \qquad |e^z| = 5$$

$$|e^z| = |e^x| = 5 \qquad X = \ln 5 = 1.609$$

$$e^x \cos y = 3, \quad e^x \sin y = 4$$

$$y = 0.927$$

$$e^z = e^{1.609 + 0.927\, i} = e^{1.609}(\cos 0.927 + i \sin 0.927)$$

$$z = 1.609 + 0.927i \pm 2n\pi i$$

Exponential Function

- It is obvious that many properties of exp z are the same as the properties of exp x with an exception in the periodicity of exp z with 2π

EXERCISES

1. Show that

   (*a*) $\exp(2 \pm 3\pi i) = -e^2$; (*b*) $\exp\left(\dfrac{2 + \pi i}{4}\right) = \sqrt{\dfrac{e}{2}}\,(1 + i)$;

   (*c*) $\exp(z + \pi i) = -\exp z$.

2. State why the function $2z^2 - 3 - ze^z + e^{-z}$ is entire.

3. Use the Cauchy–Riemann equations and the theorem in Sec. 20 to show that the function $f(z) = \exp \bar{z}$ is not analytic anywhere.

4. Show in two ways that the function $\exp(z^2)$ is entire. What is its derivative?

   Ans. $2z \exp(z^2)$.

5. Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

6. Show that $|\exp(z^2)| \leq \exp(|z|^2)$.

Thank You!