CHAPTER TWO

Electrostatics two:

- Gauss Law and Applications
- Electric Potential
- Relations between E and V
- Maxwell's Equations for Electrostatic Fields
- Dielectric Constant
- Isotropic and Homogeneous Dielectrics
- Continuity Equation
- Relaxation Time
- Poisson's and Laplace's Equations
- Boundary conditions-conductor-Dielectric and Dielectric-Dielectric
- Problems.

Gauss's Law:

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

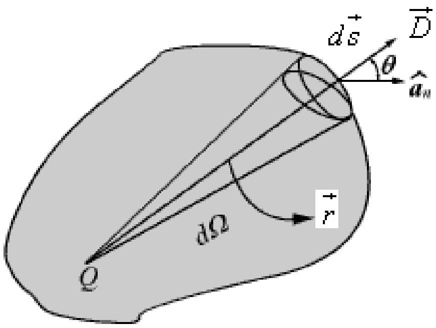


Fig 3: Gauss's Law

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon \vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds \cos\theta$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of Q.

Therefore we can write $d\psi = \frac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_S d\psi = \frac{Q}{4\pi}\oint_S d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law.

Hence we have,

$$Q_{enc} = \oint_S D \cdot ds = \int_V \rho_V \, dv$$

Applying Divergence theorem we have,

$$\oint_S D \cdot ds = \int_V \nabla \cdot D \, dv$$

Comparing the above two equations, we have

$$\int_V \nabla \cdot D \, dv = \int_V \rho_V \, dv$$

This equation is called the 1st Maxwell's equation of electrostatics.

Application of Gauss's Law:

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. $\vec{E}$ due to an infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density $_L$C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 4(a) . Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 4(b)

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E} . d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E} . d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E} . d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E} . d\vec{s}$$

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we

Can write, $\rho_L l = \varepsilon_0 E . 2\pi\rho l$

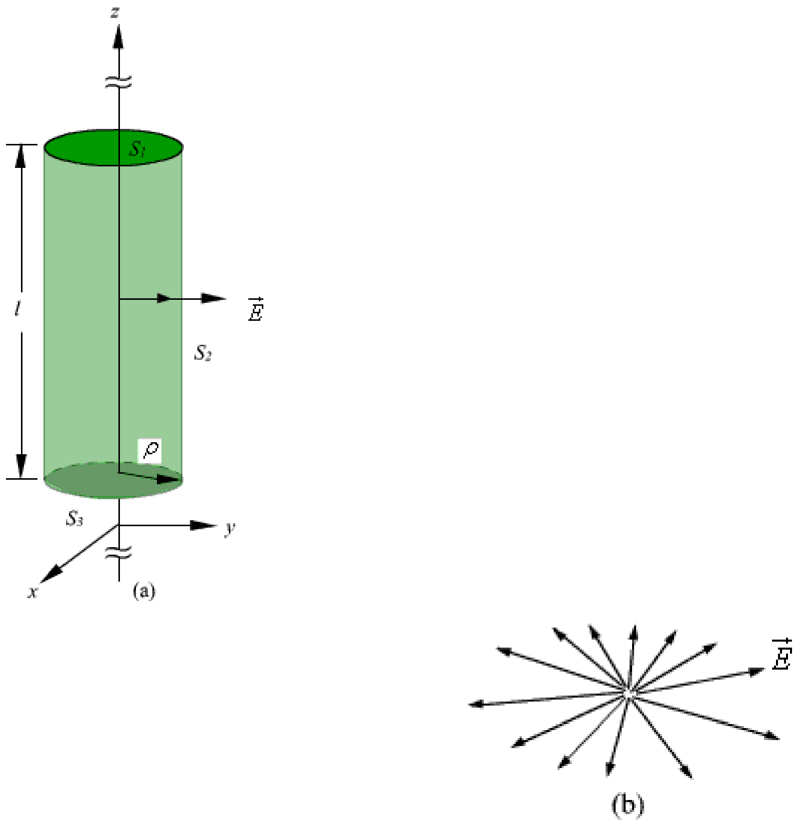


Fig 4: Infinite Line Charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x-z* plane as shown in figure 5. Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D} \cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

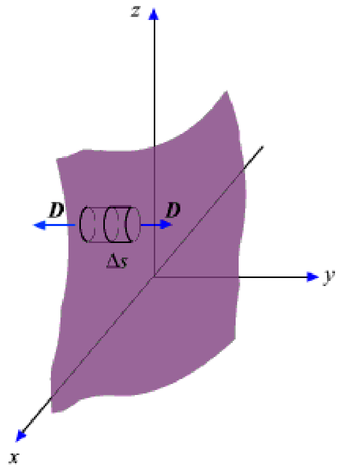


Fig 5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r0 having a uniform volume charge density of rv C/m3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius r < r0 and r > r0 as shown in Fig. 6 (a) and Fig. 6(b).

For the region $r \leq r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3$$

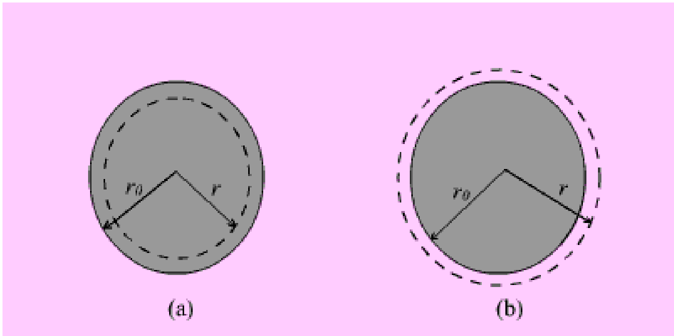


Fig 6: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$

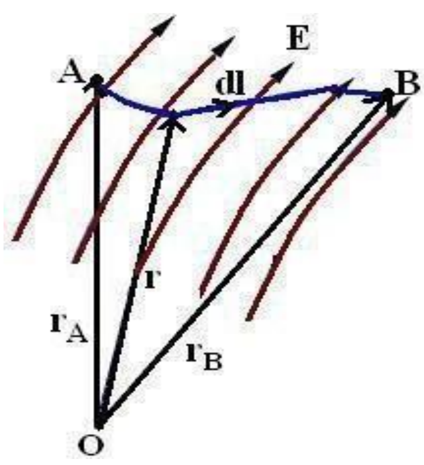
Electric Potential / Electrostatic Potential (V):

If a charge is placed in the vicinity of another charge (or in the field of another charge), it experiences a force. If a field being acted on by a force is moved from one point to another, then work is either said to be done on the system or by the system.

Say a point charge Q is moved from point A to point B in an electric field E, then the work done in moving the point charge is given as:

$W_{A\rightarrow B} = - \int AB\ (F\ .\ dl) = - Q \int AB(E\ .\ dl)$

where the – sign indicates that the work is done on the system by an external agent.



The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(V_{AB}).

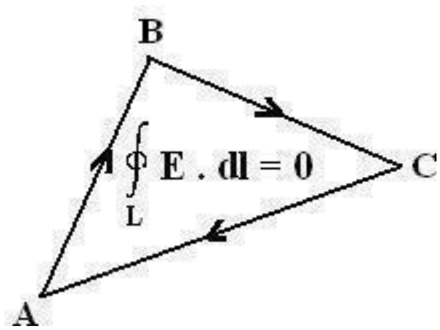
$V_{AB} = W_{A}$→B / Q

- ∫AB(E . dl)

- ∫InitialFinal (E . dl)

If the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

The electrostatic field is conservative i.e. the value of the line integral depends only on end points and is independent of the path taken.



- Since the electrostatic field is conservative, the electric potential can also be written as:

$$V_{AB} = -\int_{A}^{B} \bar{E}.\bar{dl}$$

$$V_{AB} = -\int_{A}^{p_0} \bar{E}.\bar{dl} - \int_{p_0}^{B} \bar{E}.\bar{dl}$$

$$V_{AB} = -\int_{p_0}^{B} \bar{E}.\bar{dl} + \int_{p_0}^{A} \bar{E}.\bar{dl}$$

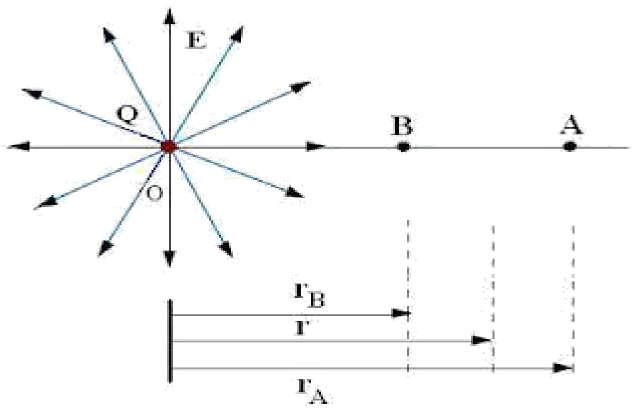
$$V_{AB} = V_B - V_A$$

Thus the potential difference between two points in an electrostatic field is a scalar field that is defined at every point in space and is independent of the path taken.

- The work done in moving a point charge from point A to point B can be written as:

$W_{A\to B} = -Q\,[V_B - V_A] = -Q\int_A^B \bar{E}\cdot d\bar{l}$

- Consider a point charge Q at origin O.



Now if a unit test charge is moved from point A to Point B, then the potential difference between them is given as:

$$V_{AB} = -\int_A^B \mathbf{E}\cdot d\mathbf{l} = -\int_{r_A}^{r_B} \mathbf{E}\cdot d\mathbf{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r\cdot dr\,\mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = V_B - V_A$$

- Electrostatic potential or Scalar Electric potential (V) at any point P is given by:

$$V = -\int_{P_0}^{P} \bar{E}\cdot d\bar{l}$$

The reference point Po is where the potential is zero (analogues to ground in a circuit). The reference is often taken to be at infinity so that the potential of a point in space is defined as

$$V = -\int_{\infty}^{P} \bar{E}\, d\bar{l}$$

Basically potential is considered to be zero at infinity. Thus potential at any point (rB = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge frominfinity to that point (i.e. rA → ∞)

Electric potential (V) at point r due to a point charge Q located at a point with position vector r1 is given as:

$$V = \frac{Q}{4\pi\varepsilon|r - r_1|}$$

Similarly for N point charges Q1, Q2 ….Qn located at points with position vectors r1, r2, r3…..rn, theelectric potential (V) at point r is given as:

$$V = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{N}\frac{Q_k}{|r - r_k|} \qquad V = \frac{Q}{4\pi\varepsilon r}$$

The charge element dQ and the total charge due to different charge distribution is given as:

$dQ = \rho_l dl$ → $Q = \int_L (\rho_l dl)$ → (Line Charge)

$dQ = \rho_s ds$ → $Q = \int_S (\rho_s ds)$ → (Surface Charge)

$dQ = \rho_v dv$ → $Q = \int_V (\rho_v dv)$ → (Volume Charge)

$$V = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon|r - r_1|} \qquad \text{(Line Charge)}$$

$$V = \int_S \frac{\rho_S\, ds}{4\pi\varepsilon|r - r_1|} \qquad \text{(Surface Charge)}$$

$$V = \int_V \frac{\rho_V\, dv}{4\pi\varepsilon|r - r_1|} \qquad \text{(Volume Charge)}$$

Second Maxwell's Equation of Electrostatics:

The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(V_{AB}).

$V_{AB} = V_B – V_A$

Similarly,

$V_{BA} = V_A – V_B$

Hence it‘s clear that potential difference is independent of the path taken. Therefore

$V_{AB} = - V_{BA}$

$V_{AB} + V_{BA} = 0$

$\int_{AB} (E \cdot dl) + [- \int_{BA} (E \cdot dl)] = 0$

$$\oint_L \mathbf{E} \cdot \mathbf{dl} = 0$$

The above equation is called the second Maxwell‘s Equation of Electrostatics in integral form..
The above equation shows that the line integral of Electric field intensity (E) along a closed path is equal to zero.
In simple words—No work is done in moving a charge along a closed path in an electrostatic field.

Applying Stokes‘ Theorem to the above Equation, we have:

$$\oint_L \mathbf{E} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{ds} = 0$$

$$\text{--->} \ \nabla \times \mathbf{E} = 0$$

If the Curl of any vector field is equal to zero, then such a vector field is called an Irrotational or Conservative Field. Hence an electrostatic field is also called a conservative field.
The above equation is called the second Maxwell‘s Equation of Electrostatics in differential form.

Relationship Between Electric Field Intensity (E) and Electric Potential (V):

Since Electric potential is a scalar quantity, hence dV (as a function of x, y and z variables) can be written as:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$\left(\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right) \cdot \left(dx\, a_x + dy\, a_y + dz\, a_z\right) = - E \,.\, dl$$

$$\nabla V \,.\, dl = - E \,.\, dl \quad \text{--->} \quad \boxed{E = -\nabla V}$$

Hence the Electric field intensity (E) is the negative gradient of Electric potential (V).
The negative sign shows that E is directed from higher to lower values of V i.e. E is opposite to the direction in which V increases.

Work Done To Assemble Charges:

In case, if we wish to assemble a number of charges in an empty system, work is required to do so. Also electrostatic energy is said to be stored in such a collection.

Let us build up a system in which we position three point charges Q_1, Q_2 and Q_3 at position r_1, r_2 and r_3 respectively in an initially empty system.

Consider a point charge Q_1 transferred from infinity to position r1 in the system. It takes no work to bring the first charge from infinity since there is no electric field to fight against (as the system is empty i.e. charge free).

Hence, $W_1 = 0$ J

Now bring in another point charge Q2 from infinity to position r2 in the system. In this case we have to do work against the electric field generated by the first charge Q1.

Hence, $W_2 = Q_2 V_{21}$

Where: V_{21} is the electrostatic potential at point r_2 due to Q_1.

- Work done W_2 is also given as:

$$W_2 = \frac{Q_2 Q_1}{4\pi\varepsilon\ |r_2 - r_1|}$$

Now bring in another point charge Q_3 from infinity to position r_3 in the system. In this case we have to do work against the electric field generated by Q_1 and Q_2.

Hence, $W_3 = Q_3 V_{31} + Q_3 V_{32} = Q_3 (V_{31} + V_{32})$

where V_{31} and V_{32} are electrostatic potential at point r_3 due to Q_1 and Q_2 respectively.

The work done is simply the sum of the work done against the electric field generated by point charge Q1 and Q2 taken in isolation:

$$W_3 = \frac{Q_3 Q_1}{4\pi\varepsilon\, |r_3 - r_1|} + \frac{Q_3 Q_2}{4\pi\varepsilon\, |r_3 - r_2|}$$

- Thus the total work done in assembling the three charges is given as:

$WE = W1 + W2 + W3$

$0 + Q2\, V21 + Q3\, (V31 + V32)$

Also total work done (WE) is given as:

$$W_E = \frac{1}{4\pi\varepsilon}\left(\frac{Q_2 Q_1}{|r_2 - r_1|} + \frac{Q_3 Q_1}{|r_3 - r_1|} + \frac{Q_3 Q_2}{|r_3 - r_2|} \right)$$

If the charges were positioned in reverse order, then the total work done in assembling them is given as:

$WE = W3 + W2+ W1$

$= 0 + Q2V23 + Q3(V12+ V13)$

Where V23 is the electrostatic potential at point r2 due to Q3 and V12 and V13 are electrostatic potential at point r1 due to Q2 and Q3 respectively.

- Adding the above two equations we have,

2WE = Q1 (V12 + V13) + Q2 (V21 + V23) + Q3 (V31 + V32)

= Q1 V1 + Q2 V2 + Q3 V3

Hence

WE =1 / 2 [Q1V1 + Q2V2 + Q3V3]

where V1, V2 and V3 are total potentials at position r1, r2 and r3 respectively.

- The result can be generalized for N point charges as:

$$W_E = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k$$

The above equation has three interpretation: This equation represents the potential energy of the system.This is the work done in bringing the static charges from infinity and assembling them in the required system. This is the kinetic energy which would be released if the system gets dissolved i.e. the charges returns back to infinity.

In place of point charge, if the system has continuous charge distribution (line, surface or volume charge), then the total work done in assembling them is given as:

$$W_E = \frac{1}{2} \int_L \rho_L V \, dl \quad \text{(Line Charge)}$$

$$W_E = \frac{1}{2} \int_S \rho_S V \, ds \quad \text{(Surface Charge)}$$

$$W_E = \frac{1}{2} \int_V \rho_V V \, dv \quad \text{(Volume Charge)}$$

Since $\rho v = \nabla \cdot D$ and $E = -\nabla V$,

Substituting the values in the above equation, work done in assembling a volume charge distribution in terms of electric field and flux density is given as:

$$W_E = \frac{1}{2}\int_V D \cdot E \, dv = \frac{1}{2}\int_V \varepsilon E^2 \, dv$$

The above equation tells us that the potential energy of a continuous charge distribution is stored in an electric field.

The electrostatic energy density wE is defined as:

$$w_E = \frac{1}{2}\varepsilon E^2 \quad ; \quad W_E = \int_V w_E \, dv$$

Properties of Materials and Steady Electric Current:

Electric field can not only exist in free space and vacuum but also in any material medium. When an electric field is applied to the material, the material will modify the electric field either by strengthening it or weakening it, depending on what kind of material it is.

Materials are classified into 3 groups based on conductivity / electrical property:

- Conductors (Metals like Copper, Aluminum, etc.) have high conductivity ($\sigma >> 1$).
- Insulators / Dielectric (Vacuum, Glass, Rubber, etc.) have low conductivity ($\sigma << 1$).
- Semiconductors (Silicon, Germanium, etc.) have intermediate conductivity.

Conductivity (σ) is a measure of the ability of the material to conduct electricity. It is the reciprocal of resistivity (ρ). Units of conductivity are Siemens/meter and mho.

The basic difference between a conductor and an insulator lies in the amount of free electrons available for conduction of current. Conductors have a large amount of free electrons where as insulators have only a few number ofelectrons for conduction of current. Most of the conductors obey ohm‘s law. Such conductors are also called ohmic conductors.

Due to the movement of free charges, several types of electric current can be caused. The different types of electric current are:

- Conduction Current.
- Convection Current.
- Displacement Current.

Electric current:

Electric current (I) defines the rate at which the net charge passes through a wire of cross sectional surface area S.

Mathematically,

If a net charge ΔQ moves across surface S in some small amount of time Δt, electric current(I) is defined as:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

How fast or how speed the charges will move depends on the nature of the material medium.

Current density:

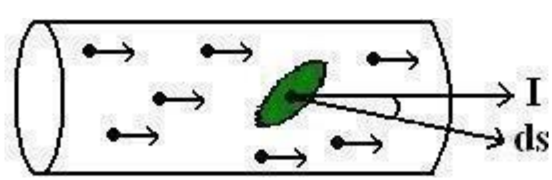
Current density (J) is defined as current ΔI flowing through surface ΔS.

Imagine surface area ΔS inside a conductor at right angles to the flow of current. As the area approaches zero, the current density at a point is defined as:

$$J = \lim_{\Delta s \to 0} \frac{\Delta I}{\Delta S}$$

The above equation is applicable only when current density (J) is normal to the surface.

In case if current density(J) is not perpendicular to the surface, consider a small area ds of the conductor at an angle θ to the flow of current as shown:



In this case current flowing through the area is given as:

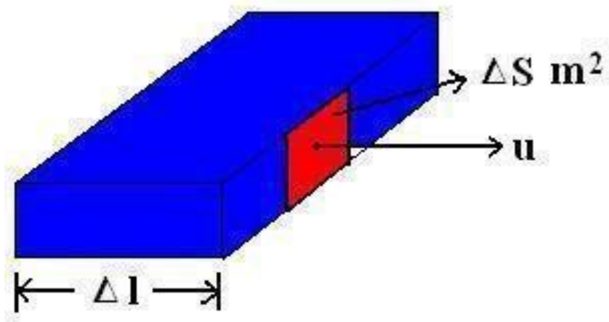
dI = J dS cosθ = J . dS and $I = \int_S \bar{J} . d\bar{s}$

Where angle θ is the angle between the normal to the area and direction of the current.

From the above equation it's clear that electric current is a scalar quantity.

CONVECTION CURRENT DENSITY:

Convection current occurs in insulators or dielectrics such as liquid, vacuum and rarified gas. Convection current results from motion of electrons or ions in an insulating medium. Since convection current doesn't involve conductors, hence it does not satisfy ohm's law. Consider a filament where there is a flow of charge ρv at a velocity u = uy ay.



- Hence the current is given as:

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

But we know $\boxed{\Delta Q = \rho_V \Delta V}$

Hence

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_V \Delta V}{\Delta t} = \rho_V \Delta S \frac{\Delta l}{\Delta t}$$

$$= \rho_V \Delta S\, u_y$$

Again, we also know that $\boxed{J_y = \frac{\Delta I}{\Delta S}}$

Hence $$J_y = \frac{\Delta I}{\Delta S} = \rho_V u_y$$

Where uy is the velocity of the moving electron or ion and ρ_v is the free volume charge density.

- Hence the convection current density in general is given as:

$$J = \rho_v u$$

Conduction Current Density:

Conduction current occurs in conductors where there are a large number of free electrons. Conduction current occurs due to the drift motion of electrons (charge carriers). Conduction current obeys ohm‘s law.

When an external electric field is applied to a metallic conductor, conduction current occurs due to the drift of electrons.

The charge inside the conductor experiences a force due to the electric field and hence should accelerate but due to continuous collision with atomic lattice, their velocity is reduced. The net effect is that the electrons moves or drifts with an average velocity called the drift velocity (υd) which is proportional to the applied electric field (E).

Hence according to Newton‘s law, if an electron with a mass m is moving in an electric field E with anaverage drift velocity υd, the the average change in momentum of the free electron must be equal to the applied force (F = - e E).

$$\frac{m \upsilon_d}{\tau} = -eE$$

where τ is the average time interval between collision.

$$\upsilon_d = \left(-\frac{e\tau}{m}\right)E$$

The drift velocity per unit applied electric field is called the mobility of electrons (μe).

υd = - μe E

where μe is defined as:

$$\mu_e = \left(-\frac{e\tau}{m}\right)$$

Consider a conducting wire in which charges subjected to an electric field are moving with drift velocity υd.
Say there are Ne free electrons per cubic meter of conductor, then the free volume charge density(ρv)within the wire is

$\rho_v = - e\ Ne$

The charge ΔQ is given as:

$\Delta Q = \rho_v\ \Delta V = - e\ Ne\ \Delta S\ \Delta l = - e\ Ne\ \Delta S\ \upsilon d\ \Delta t$

- The incremental current is thus given as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = -N_e\, e\, \Delta S\, \upsilon_d$$

Now since $\boxed{\upsilon_d = -\mu_e E}$

Therefore

$$\Delta I = N_e\, e\, \Delta S\, \mu_e E$$

The conduction current density is thus defined as:

$$J_c = \frac{\Delta I}{\Delta S} = N_e\, e\, \mu_e E = \sigma E$$

where σ is the conductivity of the material.

The above equation is known as the Ohm‘s law in point form and is valid at every point in space.

In a semiconductor, current flow is due to the movement of both electrons and holes, hence conductivity is given as:

$\sigma = (Ne\ \mu e + Nh\ \mu h)e$

DIELECTRC CONSTANT:

It is also known as Relative permittivity.

If two charges q 1 and q 2 are separated from each other by a small distance r. Then by using the coulombs law of forces the equation formed will be

$$\mathbf{F}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

In the above equation ε_0 is the electrical permittivity or you can say it, Dielectric constant.

If we repeat the above case with only one change i.e. only change in the separation medium between the charges. Here some material medium must be used. Then the equation formed will be.

$$\mathbf{F}_m = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

Now after division of above two equations

$$\frac{\mathbf{F}_o}{\mathbf{F}_m} = \frac{\varepsilon}{\varepsilon_o} = \varepsilon_r \text{ Or k}$$

In the above figure

$\varepsilon_\mathbf{r}$ is the Relative Permittivity. Again one thing to notice is that the dielectric constant is represented by the symbol (K) but permittivity by the symbol $\varepsilon_\mathbf{r}$

CONTINUITY EQUATION:

The continuity equation is derived from two of Maxwell's equations. It states that the divergence of the current density is equal to the negative rate of change of the chargedensity,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

Derivation

One of Maxwell's equations, Ampère's law, states that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking the divergence of both sides results in

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t},$$

but the divergence of a curl is zero, so that

$$\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = 0. \qquad (1)$$

Another one of Maxwell's equations, Gauss's law, states that

$$\nabla \cdot \mathbf{D} = \rho.$$

Substitute this into equation (1) to obtain

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

which is the continuity equation.

1.13 RELAXATION TIME:

- Let us consider that a charge is introduced at some interior point of a given material (conductor or dielectric).
- From, continuity of current equation, we have

$$\overline{J} = \frac{-rf_v}{rt} ----(1)$$

- We have, the point form of Ohm's law as,

$$\overline{J} = 6\overline{E} --- (2)$$

- From Gauss's law, we have,

$$\nabla\overline{D} = f_v \Rightarrow \in \nabla.\overline{E} = f_v \left[\because \overline{D} = \in .\overline{E}\right]$$

$$\therefore \nabla.\overline{E} = \frac{f_v}{\in} ------(1)$$

- Substitute equations (2) and (3) in equation (1), we get

$$\nabla.6\overline{E}f = 6.\nabla.\overline{E} = 6.\frac{f_v}{\in} = \frac{-\partial f_v}{\partial t}$$

$$\Rightarrow \frac{\partial f_v}{\partial t} + \frac{6}{\in}.f_v = 0 ------(4)$$

- The above equation is a homogeneous linear ordinary differential equation. By separating variable in eq (4), we get,

$$\frac{\partial f_v}{\partial t} = \frac{-6}{\in}.f_v$$

$$\Rightarrow \frac{\partial f_v}{\partial t} = \frac{-6}{\in}.\partial t$$

- Now integrate on both sides of above equation

$$\int \frac{\partial f_v}{\partial t} = -\frac{6}{\in}.\int \partial t$$

$$\Rightarrow \ln f_v = -\frac{6}{\in}t + \ln f_{v0}$$

Where ln p_{v0} is a constant of integration.

Thus,

$$f_v = f_{v0} e^{-t/Tr} \qquad -----(5)$$

$$T_r = \frac{\in}{6}$$

- In eq (5), f_{v0} is the initial charge density (i.e fv at t=0).
- We can see from the equation that as a result of introducing charge at some interior point of the material there is a decay of volume charge density f_v.
- The time constant "T_r" is known as the relaxation time or rearrangement time.
- Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} = 36.8 percent f its initial value.
- The relation time is short for good conductors and long for good dielectrics.

LAPLACE'S AND POISSON'S EQUATIONS:

A useful approach to the calculation of electric potentials is to relate that potential to the charge density which gives rise to it. The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

E = electric field
ρ = charge density
ε_0 = permittivity

and the electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V$$

Therefore the potential is related to the charge density by Poisson's equation

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

In a charge-free region of space, this becomes LaPlace's equation

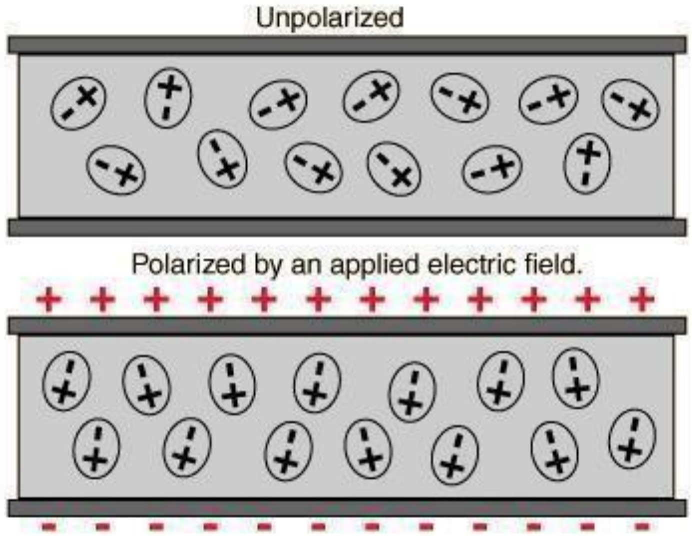
$$\nabla^2 V = 0$$

This mathematical operation, the divergence of the gradient of a function, is called the LaPlacian. Expressing the LaPlacian in different coordinate systems to take advantage of the symmetry of a charge distribution helps in the solution for the electric potential V. For example, if the charge distribution has spherical symmetry, you use the LaPlacian in spherical polar coordinates.

Since the potential is a scalar function, this approach has advantages over trying to calculate the electric field directly. Once the potential has been calculated, the electric field can be computed by taking the gradient of the potential.

Polarization of Dielectric:

If a material contains polar molecules, they will generally be in random orientations when no electric field is applied. An applied electric field will polarize the material by orienting the dipole moments of polar molecules.



This decreases the effective electric field between the plates and will increase the capacitance of the parallel plate structure. The dielectric must be a good electric insulator so as to minimize any DC leakage current through a capacitor.

The presence of the dielectric decreases the electric field produced by a given charge density.

$$E_{effective} = E - E_{polarization} = \frac{\sigma}{k\varepsilon_0}$$

The factor k by which the effective field is decreased by the polarization of the dielectric is called the dielectric constant of the material.

Solved problems:

Problem1:

Three point charges, $Q_1 = 30$ nC, $Q_2 = 150$ nC, and $Q_3 = -70$ nC, are enclosed by surface S. What net flux crosses S?

Since electric flux was defined as originating on positive charge and terminating on negative charge, part of the flux from the positive charges terminates on the negative charge.

$$\Psi_{net} = Q_{net} = 30 + 150 - 70 = 110 \text{ nC}$$

Problem-2

An electrostatic field is given by $\mathbf{E} = (x/2 + 2y)\mathbf{a}_x + 2x\mathbf{a}_y$ (V/m). Find the work done in moving a point charge $Q = -20\ \mu$C (a) from the origin to (4, 0, 0) m, and (b) from (4, 0, 0) m to (4, 2, 0) m.

(a) The first path is along the x axis, so that $d\mathbf{l} = dx\,\mathbf{a}_x$.

$$dW = -Q\mathbf{E}\cdot d\mathbf{l} = (20 \times 10^{-6})\left(\frac{x}{2} + 2y\right)dx$$

$$W = (20 \times 10^{-6})\int_0^4\left(\frac{x}{2} + 2y\right)dx = 80\ \mu\text{J}$$

(b) The second path is in the $\mathbf{a}_y$ direction, so that $d\mathbf{l} = dy\,\mathbf{a}_y$.

$$W = (20 \times 10^{-6})\int_0^2 2x\,dy = 320\ \mu\text{J}$$

Problem-3

What electric field intensity and current density correspond to a drift velocity of 6.0×10^{-4} m/s in a silver conductor?

For silver $\sigma = 61.7$ MS/m and $\mu = 5.6 \times 10^{-3}$ m²/V · s.

$$E = \frac{U}{\mu} = \frac{6.0 \times 10^{-4}}{5.6 \times 10^{-3}} = 1.07 \times 10^{-1}\ \text{V/m}$$

$$J = \sigma E = 6.61 \times 10^{6}\ \text{A/m}^2$$

Problem-4

Find the current in the circular wire shown in Fig. 6.6 if the current density is $\mathbf{J} = 15(1 - e^{-1000r})\mathbf{a}_z$ (A/m^2). The radius of the wire is 2 mm.

A cross section of the wire is chosen for S. Then

$$dI = \mathbf{J} \cdot d\mathbf{S}$$
$$= 15(1 - e^{-1000r})\mathbf{a}_z \cdot r\,dr\,d\phi\,\mathbf{a}_z$$

and

$$I = \int_0^{2\pi}\int_0^{0.002} 15(1 - e^{-1000r})r\,dr\,d\phi$$
$$= 1.33 \times 10^{-4}\text{ A} = 0.133\text{ mA}$$

Any surface S which has a perimeter that meets the outer surface of the conductor all the way around will have the same total current, $I = 0.133$ mA, crossing it.

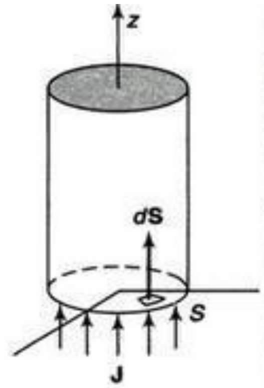


Fig. 6.6

Problem-5

Determine the relaxation time for silver, given that $\sigma = 6.17 \times 10^7$ S/m. If charge of density ρ_0 is placed within a silver block, find ρ after one, and also after five, time constants.

Since $\varepsilon \approx \varepsilon_0$,

$$\tau = \frac{\varepsilon}{\sigma} = \frac{10^{-9}36\pi}{6.17 \times 10^7} = 1.43 \times 10^{-19}\text{ s}$$

Therefore

$$\text{at } t = \tau: \quad \rho = \rho_0 e^{-1} = 0.368\rho_0$$
$$\text{at } t = 5\tau: \quad \rho = \rho_0 e^{-5} = 6.74 \times 10^{-3}\rho_0$$

Problem-6

Find the magnitudes of **D** and **P** for a dielectric material in which $E = 0.15$ MV/m and $\chi_e = 4.25$.

Since $\varepsilon_r = \chi_e + 1 = 5.25$,

$$D = \varepsilon_0\varepsilon_r E = \frac{10^{-9}}{36\pi}(5.25)(0.15 \times 10^6) = 6.96\ \mu\text{C/m}^2$$

$$P = \chi_e\varepsilon_0 E = \frac{10^{-9}}{36\pi}(4.25)(0.15 \times 10^6) = 5.64\ \mu\text{C/m}^2$$