2.1 Newton's Laws of Motion

Newton's Laws of Motion are as follows:

1. Everybody continues in its ***state of rest*** or of ***uniform motion in a straight line***, unless it is compelled by a force to change that state.
2. The change of motion is ***proportional*** to the applied force and takes place in the direction of the force.
3. To every action, there is always an equal and opposite reaction or the mutual actions of two bodies are always equal and oppositely directed.

قوانين نيوتن بالحركة هي:

1. كل جسم يستمر على حالته من السكون أو الحركة المنتظمة على خط مستقيم ، مالم تؤثر عليه قوة لتغيير تلك الحالة.
2. يتناسب تغير الحركة مع القوة المسلطة ويكون باتجاه تاثيرالقوة.
3. لكل فعل، يكون هناك دائمًا رد فعل مساو له بالمقداروممعاكس بالاتجاه. الافعال المتبادلة لجسمين تكون دائما متساوية معاكسة بالاتجاه.

2.2 Newton's First Law: Inertial Reference Systems

The first law describes a common property of matter, namely, inertia.

Inertia is the resistance of all matter to having its motion changed

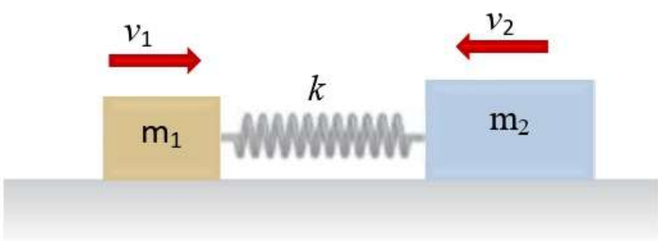
يصف القانون الأول خاصية مشتركة للمادة ، وهي القصور الذاتي.

القصور الذاتي هو مقاومة الجسم لتغير حركته

2.3 Mass and Force: Newton's Second and Third Law

Consider two masses $m_1$and m_2 attached by a spring and they initially were at rest. If the two masses were pushed together, compressing the spring and then releasing them, so that they fly apart attaining speeds $\vec{v}_1$and $\vec{v}_2$ respectively.

نفترض وجود كتلتين m_1 و m_2 مرتبطتين بنابض حلزوني وكانا في البداية في حالة سكون. إذا تم دفع الكتلتين باتجاه بعضهما ، عنها ينضغط النابض، ثم نتركهما بحيث ينفصلان عن بعضهما بسرع $\vec{v}_1$ و $\vec{v}_2$ على التوالي وباتجاهين متعاكسين.



The ratio of the two masses:

$$\frac{m_2}{m_1} = \left|\frac{\vec{v}_1}{\vec{v}_2}\right| \qquad \dots\dots(1)$$

Eqn. (1) is equivalent to:

$$\Delta(m_1\vec{v}_1) = -\Delta(m_2\vec{v}_2) \qquad \dots\dots(2)$$

because the ***initial*** velocities of each mass are ***zero*** and the final velocities $\vec{v}_1$ and $\vec{v}_2$ are in ***opposite*** directions. If we divide by Δt and take limits as $\Delta t \to 0$ obtain:

$$\frac{d}{dt}(m_1\vec{v}_1) = -\frac{d}{dt}(m_2\vec{v}_2) \quad \dots\dots\dots.(3)$$

The product of mass and velocity, $m\vec{v}$, is called ***linear momentum.***

So the second law can be rephrased as follows: ***The time rate of change of an object's linear momentum is proportional to the impressed force***, ***F***. Thus, the second law can be written as:

يُطلق على حاصل ضرب الكتلة والسرعة الزخم الخطي.

لذلك يمكن إعادة صياغة القانون الثاني للحركة على النحو التالي: القوة تساوي المعدل الزمني لتغير الزخم الخطي للجسم .

$$\vec{F} \propto \frac{d}{dt}(m\vec{v})$$

$$\vec{F} = k\frac{d}{dt}(m\vec{v})$$

where k is a constant of proportionality. Let $k = 1$

$$\therefore \vec{F} = \frac{d}{dt}(m\vec{v})$$

where m constant, finally express Newton's second law in the familiar form:

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \qquad \dots..(4)$$

$\vec{F}$: is the net force acting upon the mass m; that is, it is the vector sum of all of the individual forces acting upon m.

From Eqn. (3)

$$\vec{F}_1 = -\vec{F}_2 \qquad \dots\dots.(5) \qquad \textbf{\textit{Newton's third law}}$$

Two interacting bodies exert equal and opposite force upon one another.

كل جسمين يؤثر احدهما على الاخر بقوة متساوية بالمقدار ومتعاكسة بالاتجاه

2.4 Linear Momentum

$\vec{P} = m\vec{v}$ ……(1)

$\vec{F} = \frac{d\vec{P}}{dt}$ ……(2)

Sub Eqn. (2) in third Newton's law $\vec{F}_1 = -\vec{F}_2$

$\frac{d\vec{P}_1}{dt} = -\frac{d\vec{P}_2}{dt}$

$\frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = 0$ *or* $\frac{d}{dt}(\vec{P}_1 + \vec{P}_2) = 0$

$\therefore$ $\vec{P}_1 + \vec{P}_2 = \text{constant}$ (***conservation of linear momentum***)

∴ Newton's third law implies that the total momentum of two mutually interacting bodies is a constant.

The equation of motion for a particle subject to the influence of a net force, $\vec{F}$, writing as the vector sum of all the forces acting on the particle.

$\vec{F} = \sum \vec{F}_i = m\,\frac{d^2\vec{r}}{dt^2} = m\vec{a}$

كل جسمين يؤثر احدهما على الاخر بقوة متساوية ومعاكسة بالاتجاه

Example:

A spaceship of mass M is traveling in deep space with initial velocity ($\vec{v}_i = 20\ km/s$) relative to the sun. It ejects a rear stage of mass ($0.2\,M$) with speed ($\vec{u} = 5\ km/s$), find the final velocity $\vec{v}_f$ of the space ship after ejection.

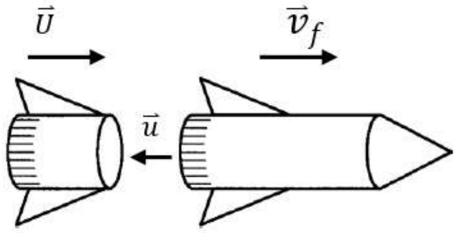
Solution:

The system of spaceship plus rear stage is a closed system upon which no external forces act; the total linear momentum is conserved.

$\Delta\vec{P} = \vec{P}_f - \vec{P}_i = 0$

نظام سفينة الفضاء اضافة الى المرحلة الخلفية هو نظام مغلق ليس عليه قوى خارجية ؛ عليه يكون الزخم الخطي الكلي محفوظ

$\therefore \vec{P}_f = \vec{P}_i$ ……(1)



$\vec{P}_i$ = initial linear momentum

$\vec{P}_f$ = final linear momentum

Let U be the velocity of the rear stage after ejection.

$\vec{P}_i = M\vec{v}_i$ ……(2)

The total momentum of the system after ejection is then:

$\vec{P}_f = 0.2M\,\vec{U} + 0.8\,M\vec{v}_f$ …… (3)   الزخم الخطي الكلي للنظام بعد الانفصال

$\vec{u} = \vec{v}_f - \vec{U}$

$\vec{U} = \vec{v}_f - \vec{u}$ …… (4)

Sub. Eqn. (4) in Eqn. (3)

$\vec{P}_f = 0.2M(\vec{v}_f - \vec{u}) + 0.8\,M\vec{v}_f$ …… (5)

Eqn. (5) equal Eqn.(2), so:

$[0.2M(\vec{v}_f - \vec{u}) + 0.8\,M\vec{v}_f = M\,\vec{v}_i\,] \div M$

$0.2\vec{v}_f + 0.8\vec{v}_f = \vec{v}_i + 0.2\vec{u}$

$\vec{v}_f = \vec{v}_i + 0.2\,\vec{u}$

$= 20\,km/s + 0.2(5\,km/s)$

$\vec{v}_f = 21\,km/s$

2.5 Rectilinear Motion

When a moving particle remains on a ***single straight line***, the motion is said to be "***rectilinear*** ". The general equation motion is:

$\vec{F}\,(x, \dot{x}\,, t) = m\ddot{x} = m\vec{a}$

عندما يستمر الجسم في حركته على خط مستقيم واحد، يُقال إن الحركة "مستقيمة"

Note: We usually use the single variable *x* to represent the position of a particle. To avoid unnecessary use of subscripts, we often use the symbols $v, a, \dot{x},\ \ddot{x}$ and F respectively, rather than $v_x, a_x, \dot{x}_x,\ \ddot{x}_x$ and F_x.

Special Cases:

1. ***Constant Force***

$\vec{F} = constant\ \ then\ \vec{a} = constant$

$\therefore \vec{F} = m\frac{d\vec{v}}{dt} \rightarrow \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} = \vec{a} = constant$ …… (1)

$m\ddot{x} = m\vec{a}$

$\vec{F} = \ddot{x} = \frac{d\vec{v}}{dt} = \vec{a}$

$d\vec{v} = \vec{a}dt$

$\int_{u_0}^{u} d\vec{v} = \int_0^t \vec{a}\, dt$

$v - v_0 = at$ $v_0 \equiv$ Initial velocity

$\vec{v} = at + v_0$ ……(2)

$\therefore \vec{v} = \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = at + v_0$

$\int_{x_0}^{x} dx = \int_0^t (at + v_0)\, dt$

$x - x_0 = \frac{1}{2}at^2 + v_0 t$

$x = \frac{1}{2}\, at^2 + v_0 t + x_0$ ……(3)

$at = v - v_0$

$\therefore t = \frac{v - v_0}{a}$ ……(4)

$x - x_0 = \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 + v_0\left(\frac{v - v_0}{a}\right)$

$2a\,(x - x_0) = a^2\left(\frac{v - v^2}{a}\right)^2 + 2v_0(v - v_0)$

$2a\,(x - x_0) = v^2 + v_0^2 - 2vv_0 + 2vv_0 - 2v_0^2$

$2a\,(x - x_0) = v^2 - v_0^2$ …..(5)

The equations of ***uniformly accelerated motion***

$$\vec{v} = at + v_0$$
$$x = \frac{1}{2}\, at^2 + v_0 t + x_0$$
$$2a\,(x - x_0) = v^2 - v_0^2$$

2. *Free Fall*

In the case of a body ***falling freely*** near the surface of the Earth, neglecting air resistance, the acceleration is very nearly constant

في حالة السقوط الحر لجسم ما بالقرب من سطح الأرض، وإهمال مقاومة الهواء، يكون التعجيل ثابتًا

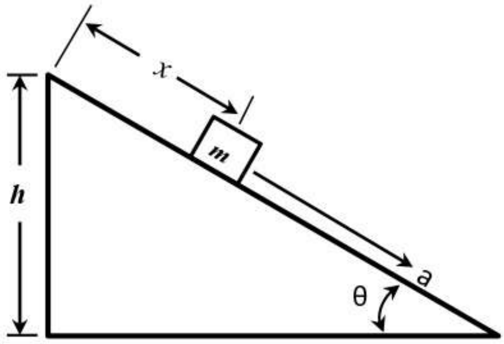
$a = g = 9.8\ \frac{m}{sec^2} = 32\ ft\,/sec^2$

$\vec{F} = m\vec{g}$

Example:

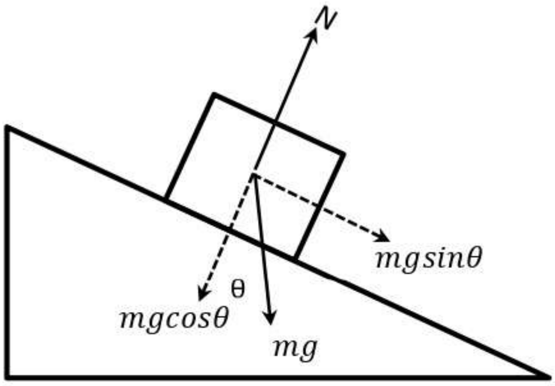
A block is sliding down on a smooth plane inclined at angle $\boldsymbol{\theta}$ to horizontal. If the height of the plane is h as shown in the figure and the block is released from rest $(v_0 = 0)$ at the top, what will be its speed when it reaches the bottom? Then how the accelerate will become when surface is not smooth?

جسم ينزلق اسفل سطح املس يميل بزاوية θ عن الافق. اذا كان ارتفاع السطح هو h كما هو موضح في الشكل وتم تحرير الجسم من السكون في الأعلى ، ما هي سرعته عندما يصل إلى اسفل السطح؟ وكم سيصبح تعجيله اذا كان السطح خشن



Solution:

a)*Smooth plane* (*No Frictional force*)



$\vec{F} = m\vec{a}$

$\vec{a} = \frac{\vec{F}}{m}$

$\therefore\ \vec{F} = mg\sin\theta$

$m\vec{a} = mg\sin\theta$

$\therefore \vec{a} = g\sin\theta \ldots\ldots (1)$

$x - x_0 = \frac{h}{\sin\theta} \ldots\ldots (2)$

Using one of the equation of motion $(2a\,(x - x_0) = (v^2 - v_0^2))$

where $v_0 = 0$

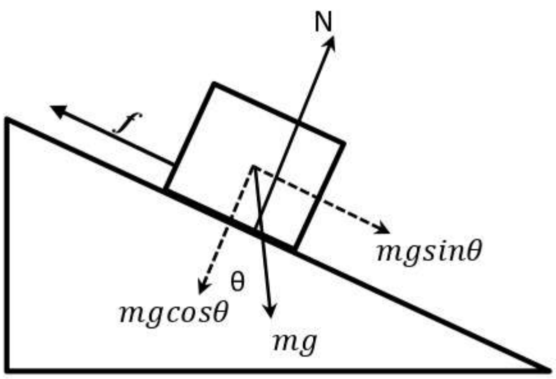
$\therefore v^2 = 2a(x - x_0) \ldots\ldots (3)$

$v^2 = 2(g\sin\theta)\left(\frac{h}{\sin\theta}\right)$

$\therefore v^2 = 2gh$

$v = \sqrt{2gh} \ldots\ldots (4)$

b) *Rough Plane* (*Frictional force*)



$\vec{F} = mg\sin\theta - f \ldots\ldots (5)$

$\vec{f}\,\alpha\,\vec{N}$

$\vec{f} = \mu\vec{N}$

N: normal force, μ: coefficient of sliding or kinetic friction

From figure:

$$N = mg\cos\theta$$

$$\therefore f = \mu mg\cos\theta \qquad \dots\dots(6)$$

$$\vec{F} = m\vec{a} = mg\sin\theta - \mu mg\cos\theta$$

$$\vec{a} = g\,(\sin\theta - \mu\cos\theta)$$

For motion up the plane, the direction of the frictional force is reversed; that is, it is in the positive x direction. The acceleration (actually ***deceleration***) is then:

$$\vec{a} = g(\sin\theta + \mu\cos\theta)$$

للحركة اعلى السطح، يتم عكس اتجاه قوة الاحتكاك. اي تكون في الاتجاه الموجب. التسارع (في الواقع هنا هو تباطؤ)

2.6 Forces that Depend on Position
(The Concepts of Kinetic and Potential Energy)

مفهوم الطاقة الحركية والطاقة الكامنة

- Force depends only on the particles position
- Electrostatic and gravitational forces.
- Forces of elastic tension or compression.

- قوة تعتمد على موضع الجسيمات فقط
- القوى الكهروستاتيكية والجاذبية.
- قوى التوتر المرن أو الضغط.

If the force is *independent of velocity or time*, then the differential equation for rectilinear motion is simply:

$$\vec{F}(x) = m\ddot{x} \qquad \dots\dots(1)$$

Using the chain rule

$$\therefore \ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dt}\frac{dx}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx}$$

$$\therefore \ddot{x} = v\frac{dv}{dx} \qquad \dots\dots(2)$$

Sub. Eqn. (2) in Eqn. (1)

$$\therefore F(x) = mv\frac{dv}{dx} \qquad \dots\dots(3)$$

Also, Eqn.(3) may be written as:

$$\vec{F}(x) = \frac{m}{2}\frac{d(v^2)}{dx}$$

$$\vec{F}(x) = \frac{dT}{dx} \qquad \dots\dots(4)$$

where $\left[T = \frac{1}{2}mv^2\right] \rightarrow$ Kinetic energy of the particle

$$F(x)dx = dT$$

$$\int_{x_0}^{x} F(x)dx = \int_{T_0}^{T} dT = T_x - T_0 = W \dots\dots (5)$$

The ***work*** is equal to the change in the kinetic energy of the particle.

Let us define a function $V(x)$ such that

الشغل يساوي التغير في الطاقة الحركية للجسم

$$F(x) = -\frac{dV}{dx}$$

$$\int_{x_0}^{x} F(x)\, dx = -V + constant \quad \dots\dots (6)$$

V: ***Potential energy***.

The function $V(x)$ is called the potential energy in terms of (x), the work integral is

$$W = \int_{x_0}^{x} f(x)\, dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

$$-V + costant = T$$

$$T + V = costant$$

$$\frac{1}{2} mv^2 + V(x) = costant = E \quad \dots\dots(7) \qquad \textbf{Total energy equation}$$

Total energy (total mechanical energy) it is equal to the sum of the kinetic and potential energies and is constant throughout the motion of the particle.

Such force (depend on position only) called ***Conservative force***. ***Nonconservative forces*** that is, those for which no potential energy function exists are usually of a dissipational nature, such as friction.

(***Free Fall***) (Constant acceleration) is an example of ***conservative motion***.

إجمالي الطاقة يساوي مجموع الطاقات الحركية والاحتمالية وهو ثابت طوال حركة الجسيم. ان هذه القوة (تعتمد على الموضع فقط) والتي تسمى القوة المحافظة. عادة ما تكون القوى غير المحافظة ، تلك التي لا توجد لها دالة طاقة كامنة ، فتكون اعتياديا من نوع التبديد، مثل الاحتكاك. في حين ان السقوط الحر (تعجيل ثابت) هو مثال على الحركة المحافظة.

$$\frac{1}{2} mv^2 + V(x) = E$$

$$\frac{1}{2} mv^2 = E - V(x) \qquad * \frac{2}{m}$$

$$v^2 = \frac{2}{m}[E - V(x)]$$

$$\therefore \vec{v} = \mp\sqrt{\frac{2}{m}[E - V(x)]} \quad \dots\dots(8)$$ ***Equation of velocity as a function of (x)***

$$\therefore \vec{v} = \frac{dx}{dt} = \mp\sqrt{\frac{2}{m}[E - V(x)]}$$

$$dt = \mp\frac{dx}{\sqrt{\frac{2}{m}[E - V(x)]}}$$

$t = \int_{x_0}^{x} \frac{\mp dx}{\sqrt{\frac{2}{m}[E - V(x)]}}$ ……(9) ***Equation of time (t) as a function of (x)***

Note that:

1. When $V(x) \leq E \rightarrow The\ \boldsymbol{velocity}\ (v)\ is\ \boldsymbol{real}$
2. When $V\ (x) = E\ \ \rightarrow The\ \boldsymbol{velocity}\ (v) = 0$
   This means that the particle must come to rest and reverse its motion at points for which the equality holds. These points are called the ***turning points*** of the motion.
3. When $V(x) \geq E\ \ \rightarrow The\ \boldsymbol{velocity}\ (v)$ is $\boldsymbol{imaginary}$

- تكون السرعة (v) حقيقية في حالة كون $V(x)$ اقل او مساوية للطاقة الكلية للجسم.
- تكون السرعة مساوية للصفر في حالة كون $V(x)$ مساوية للطاقة الكلية وهذا يعني أن الجسيم يجب أن يقف ويعكس حركته عند نقاط تحمل المساواة فيها. وتسمى هذه النقاط تسمى نقاط الرجوع للحركة.
- تكون السرعة ذات قيمة خيالية في حالة كون $V(x)$ مساوية للطاقة الكلية للجسم.

Example: (*Free Fall*)

A body is projected upward in the positive *x*-direction with initial speed (v_0). choosing $x = 0$ as initial point of projection , find the maximum height attained by the body and then find the equation of time (t) in terms of (g)

Solution:

Choose the *x* direction to be positive upward, and then the gravitational force is equal to $(-mg)$.

At $x = 0 \quad \dot{x} = v_0$

At $x = max \quad \dot{x} = ?$

$F = -mg \ldots\ldots (1)$

$F = -\frac{dV(x)}{dx}$

$\therefore \int F(x)dx = -V + c \quad \ldots\ldots (2)$

Sub Eqn. (1) in Eqn. (2)

$\therefore \int -mg\ dx = -V + c$

$-mgx = -V + c$

choose $c = 0$

then $V = 0$ at $x = 0$

$\therefore V = mgx \ldots\ldots (3)$ ***Potential energy***

$E = \frac{1}{2}\ mv^2 + V \ldots\ldots (4)$

Sub Eqn. (3) in Eqn. (4)

$\therefore E = \frac{1}{2}mv^2 + mgx \quad \ldots\ldots(5)$

The body be projected upward with ***initial speed*** $\boldsymbol{v_0}$ from the ***origin*** $\boldsymbol{x = 0}$

$E = \frac{1}{2}mv_0^2 + mg(0)$

$\therefore E = \frac{1}{2}\,mv_0^2 \quad \ldots\ldots(6)$

Energy equation during body motion

$\therefore\ \frac{1}{2}\,mv_0^2 = \frac{1}{2}mv^2 + mgx \quad \ldots\ldots(7)$

$v^2 = v_0^2 - 2gx \quad \ldots.\ldots\ (8)$

The ***turning point*** of the motion, which is in this case the maximum height $(X = X_{max})$, is given by setting $v = 0$.

نقطة الرجوع في الحركة، تكون في هذه الحالة عند أقصى ارتفاع

$0 = v_0^2 - 2gX_{max}$

$\frac{1}{2}\,m\,v_0^2 = mgX_{max}$

$\therefore X_{max} = \frac{v_0^2}{2g}$ →The maximum height that the body attained

To obtain ***time*** $\boldsymbol{(t)}$ from Eqn. (8)

$v^2 = v_0^2 - 2gx$

$v^2 = \left(\frac{dx}{dt}\right)^2 = v_0^2 - 2gx$

$\frac{dx^2}{dt^2} = v_0^2 - 2gx$

$dt^2 = \frac{dx^2}{v_0^2 - 2gx}$

$dt^2 = (v_0^2 - 2gx)^{-1}dx^2$

$dt = (v_0^2 - 2gx)^{-1/2}dx$

By integration two side:

$\int_0^t dt = \int_0^x (v_0^2 - 2gx)^{-1/2}\,dx$

$t = -\left.\frac{2(v_0^2 - 2gx)^{1/2}}{2g}\right|_0^x = -\frac{(v_0^2 - 2gx)^{1/2}}{g} + \frac{v_0}{g}$

$t = \frac{v_0}{g} - \frac{(v_0^2 - 2gx)^{1/2}}{g}$

2.7 Variation of Gravity with Height

We assumed that g was constant. Actually, the force of gravity between two particles is inversely proportional to the square of the distance between them (***Newton's law of gravity***). The ***gravitational force*** that the ***Earth exerts on a body of mass m*** is given by:

افترضنا أن g ثابت في الواقع، إن قوة الجاذبية بين جسمين تتناسب عكسيا مع مربع المسافة بينهما (قانون الجذب العام لنيوتن).

$$F_r = -G\frac{Mm}{r^2} \quad ....(1)$$

Where G: is Newton's constant of gravitation

M: is the mass of the Earth

r: is the distance from the center of the Earth to the body.

We know that there is a relation between the force and potential energy:

$$F = -\frac{\partial V}{\partial r} \rightarrow \partial V = -F\partial r \rightarrow dV = -Fdr \rightarrow F$$

$$\int dV = -\int -\frac{GMm}{r^2}dr$$

$$V(r) = GMm\left(\frac{r^{-2+1}}{-1}\right)$$

$$V(r) = -\frac{GMm}{r} \quad ....(2)$$ ***Potential Energy function***

If we ***neglect air resistance,*** the differential equation of motion is:

في حالة اهمال مقاومة الهواء

$$m\ddot{r} = -G\frac{Mm}{r^2} \quad ......(3)$$

$$\ddot{r} = \frac{d\dot{r}}{dt} \quad * \frac{dr}{dr}$$

$$\ddot{r} = \frac{d\dot{r}}{dt}\frac{dr}{dr} = \frac{d\dot{r}}{dr}\frac{dr}{dt}$$

$$\therefore \ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad ......(4)$$

Sub. Eqn. (4) in Eqn.(3)

$$m\dot{r}\frac{d\dot{r}}{dr} = -G\frac{Mm}{r^2}$$

Integrating both side with respect $\dot{r}$ and r

$$m\int \dot{r}d\dot{r} = -GMm\int\frac{dr}{r^2}$$

$$m\int \dot{r}d\dot{r} = -GMm\int r^{-2}\,dr$$

$\frac{1}{2}m\dot{r}^2 = -\frac{GMm}{-1}r^{-2+1} = GMmr^{-1} + c$

$\frac{1}{2}m\dot{r}^2 = \frac{GMm}{r} + c$

$\frac{1}{2}m\dot{r}^2 = \frac{GMm}{r} + c \quad \dots\dots(5)$

$c = E$ is constant of integration.

$\therefore \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = E \quad \dots\dots(6)$ ***Free Falling Energy equation***

Eqn. (6) is ***energy equation***, represent the sum of the ***kinetic energy*** (1st term) and the ***potential energy*** (2nd term) remain constant throughout the motion of a ***falling body***.

المعادلة رقم (6) تمثل معادلة الطاقة، وتمثل مجموع الطاقة الحركية (الحد الأول) والطاقة الكامنة (الحد الثاني) والتي تبقى ثابتة طوال حركة الجسم الساقط.

When the projectile shot upward from the surface of the earth with initial speed v_0:

$\frac{1}{2}mv_0^2 - \frac{GMm}{r_e} = E \quad \dots\dots(7)$

Where r_e is the ***radius of the earth***

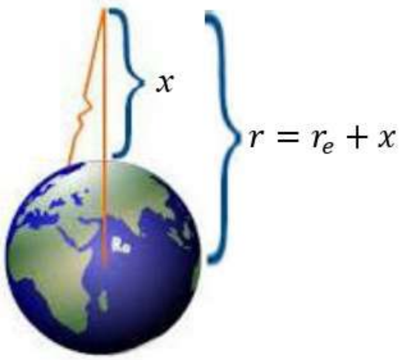
Now, in order to find the speed of the projectile at any height x above the earth's surface, combining the last two energy equations (6) and (7):

الآن، لايجاد سرعة القذيفة في أي ارتفاع x فوق سطح الأرض، نجمع المعادلتين الأخيرتين للطاقة (6) و (7)

$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = \frac{1}{2}mv_0^2 - \frac{GMm}{r_e}$

$\frac{1}{2}mv_0^2 - \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r} - \frac{GMm}{r_e} = 0$

$\frac{1}{2}m(v_0^2 - \dot{r}^2) + GMm(\frac{1}{r} - \frac{1}{r_e}) = 0$



Substituting by ($v = \dot{r}$), then multiply the Eqn. by ($\frac{2}{m}$), we get:

$(v_0^2 - v^2) + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right) = 0$

$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right)$

But $r = r_e + x$

$\therefore v^2 = v_0^2 + 2GM\left(\frac{1}{r_e+x} - \frac{1}{r_e}\right)$ ..... (8) ***Speed at any height above the earth surface***

Now the equation of gravity acceleration for the projectile on the earth's surface is given: (*gravitational force equal to weight of body*)

الآن يتم إعطاء معادلة تعجيل الجاذبية للقذيفة على سطح الأرض: (قوة الجاذبية تساوي وزن الجسم)

$$-\frac{GMm}{r_e^2} = -mg$$

$$g = \frac{GM}{r_e^2} \quad \text{......} (9)$$

$$G = \frac{gr_e^2}{M} \quad \text{......} (10)$$

Sub. Eqn. (10) in Eqn. (8)

$$v^2 = v_0^2 + \frac{2Mgr_e^2}{M}\left(\frac{1}{r_e+x} - \frac{1}{r_e}\right)$$

$$v^2 = v_0^2 + 2gr_e^2\left(\frac{r_e-(r_e+x)}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 + 2gr_e^2\left(\frac{r_e- r_e-x}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 + 2gr_e^2\left(\frac{-x}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 - 2gx\left(\frac{r_e^2}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 - 2gx\left(\frac{r_e}{r_e+x}\right)$$

$$v^2 = v_0^2 - 2gx\left[\frac{\frac{r_e}{r_e}}{\left(\frac{r_e}{r_e}+\frac{x}{r_e}\right)}\right] = v_0^2 - 2gx\left[\frac{1}{1+\frac{x}{r_e}}\right]$$

$v^2 = v_0^2 - 2gx(1+\frac{x}{r_e})^{-1}$.....(11) ***Speed of projectile with variant gravity acceleration***

When $x \ll re \rightarrow \left(\frac{x}{r_e}\right)$ can be neglected, then Eqn. (11) reduces to the form:

$v^2 = v_0^2 - 2gx$ ......(12) ***Speed of projectile with uniform gravitational field***

The ***maximum height*** (***turning point***) is found by setting $v = 0$ and solving for x

$$\therefore 0 = v_0^2 - 2gx_{max}(1+\frac{x_{max}}{r_e})^{-1}$$

$$v_0^2 = 2gx_{max}(1+\frac{x_{max}}{r_e})^{-1}$$

$$x_{max} = \frac{v_0^2}{2g}(1 + \frac{x_{max}}{r_e})$$

$$x_{max} = \frac{v_0^2}{2g} + \frac{v_0^2}{2g}\frac{x_{max}}{r_e}$$

$$x_{max} - \frac{v_0^2}{2g}\frac{x_{max}}{r_e} = \frac{v_0^2}{2g}$$

$$x_{max}(1 - \frac{v_0^2}{2gr_e}) = \frac{v_0^2}{2g}$$

$$x_{max} = \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2gr_e}\right)^{-1}$$ …..(13) ***Maximum height of the projectile***

Again, if $v_0^2 \ll 2gr_e \;\rightarrow\; \frac{v_0^2}{2gr_e}$ *can be neglected*

$$x_{max} = h = \frac{v_0^2}{2g}$$ ……..(14) ***Maximum height of the projectile with low initial speed***

To find v_0 that make the projectile escape from the earth's gravity, which is called ***escape speed***, we need to expand the series in Eqn. (13), by using binomial, as in:

لايجاد v_0 التي تجعل الجسم المقذوف يهرب من جاذبية الأرض، والتي تسمى سرعة الهروب، نحتاج إلى ايجاد مفكوك المتسلسلة في معادلة (13)، باستخدام متعدد الحدود.

$$(1 - \frac{v_0^2}{2gr_e})^{-1} = \left(1 - \frac{v_0^2}{2gr_e} + \left(\frac{v_0^2}{2gr_e}\right)^2 - \cdots\right)$$

Sub. in Eqn. (13)

$$x_{max} = h = \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2gr_e} + \left(\frac{v_0^2}{2gr_e}\right)^2 - \cdots\right)$$

$$x_{max} = h = \frac{v_0^2}{2g} - \left(\frac{v_0^2}{2g}\right)^2\frac{1}{r_e} + \left(\frac{v_0^2}{2g}\right)^3\frac{1}{r_e^2} + \cdots$$

Neglecting high terms

$$x_{max} = h = \frac{v_0^2}{2g}$$

$$v_0^2 = 2gh$$

$$v_0 = \sqrt{2gh}$$

$h = x_{max} = r_e = 6.4 \times 10^6 m$ and $g = 9.8\,\mathrm{m}/s^2$

$$\therefore v_e = \sqrt{2g\, r_e} \cong 11km/s$$ ……(15) ***Escape velocity***

In the ***Earth's atmosphere***, the average speed of air molecules (O_2 and N_2) is about 0.5 *km/s*, which is considerably less than the escape speed, so the Earth retains its atmosphere. The ***moon***, has no atmosphere; because the escape speed at the moon's surface, owing to the moon's small mass, is considerably smaller than that at the Earth's surface, any oxygen or nitrogen would eventually disappear.

في الغلاف الجوي للأرض ، يبلغ متوسط سرعة جزيئات الهواء (O_2 و N_2) حوالي 0.5 كم / ثانية ، وهو أقل بكثير من سرعة الهروب، لذلك تحتفظ الأرض بجوها. القمر ليس له جو، لأن سرعة الهروب عند سطح القمر، بسبب كتلة القمر الصغيرة، أصغر بكثير من سرعة الهروب عند سطح الأرض، فإن أي أوكسجين أو نيتروجين سيختفي في النهاية.

2.8 The Force as a Function of Velocity Only
(*Horizontal Motion with Liner Resistance*)

It often happens that the force that acts on a body is a *function of the velocity of the body*. This is true, for example, in the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of *v* only, the differential equation of motion may be written in either of the two forms

غالبا ما يحدث أن القوة المؤثرة على الجسم هي دالة لسرعة الجسم. هذا صحيح، على سبيل المثال، في حالة مقاومة اللزوجة التي تؤثرعلى الجسم المتحرك عبر مائع. إذا أمكن التعبير عن القوة كدالة للسرعة فقط ، فيمكن كتابة المعادلة التفاضلية للحركة في صيغيتين

$F_0 + F(v) = m\frac{dv}{dt} \ldots\ldots (1)$

$F_0 =$ is constant force that $\boldsymbol{dose\ not\ depend\ on\ v}$

$mdv = F(v)dt$

$dt = \frac{m\,dv}{F(v)}$

By integrating both sides:

$dt = \int \frac{m\,dv}{F(v)} \Longrightarrow t \to t(v) \ldots\ldots (2)$

Assuming that we can solve the above Eqn. for (v)

$(v) = v(t)$

Second integration: $\int v(t)dt = x(t)$

$$\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$$

Sub. in Eqn. (1)

$$F_0+F(v)=mv\frac{dv}{dx}$$

$$dx=\frac{mv\,dv}{F(v)}$$

$$x=\int\frac{mv\,dv}{F(v)}=x(v) \qquad \ldots\ldots(3)$$ ***Position as function of*** (v)

Example: (*Horizontal Motion with Liner Resistance)*

A block is projected with initial velocity (v_0) on a smooth horizontal plane, but it was affected by air resistance proportional of (v) i.e. $F(v)=-cv$. Find the equation of time (t) as a function of (v), then find the equation of velocity and displacement as a function of (t).

Solution:

$$F=-cv\ \ldots\ldots(1)$$

$$F=m\frac{dv}{dt}\ \ldots\ldots(2)$$

$$-cv=m\frac{dv}{dt}$$

$$dt=-\frac{m}{c}\frac{dv}{v}$$

By integrating both sides

$$\int_0^t dt=-\frac{m}{c}\int_{v_0}^{v}\frac{dv}{v}$$

$$t=-\frac{m}{c}\,ln\,v\Big]_{v_0}^{v}=-\frac{m}{c}(\ln v-\ln v_0)$$

$$t=-\frac{m}{c}\ln\left(\frac{v}{v_0}\right) \qquad \ldots\ldots..(3)$$ ***Equation of time as a function of*** (v)

Multiplying by $\left(\frac{-c}{m}\right)$

$$\frac{-c\,t}{m}=\ln\frac{v}{v_0}$$

$$e^{\frac{-c\,t}{m}}=\frac{v}{v_0}$$

$$v=v_0\ e^{\frac{-c\,t}{m}} \qquad \ldots\ldots(4)$$ ***Velocity as a function of*** (t)

$$v = \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = v_0\, e^{\frac{-c\,t}{m}}$$

$$\int_0^x dx = \int_0^x v_0\, e^{\frac{-c\,t}{m}}\, dt \ldots\ldots (5)$$

$$x = \int_0^t v_0\, e^{\frac{-c\,t}{m}}\ dt * \frac{-\frac{c}{m}}{-\frac{c}{m}}$$

$$x = -\ \frac{m\,v_0}{c}\int_0^t \frac{-c}{m}\ e^{\frac{-c\,t}{m}} dt$$

$$x = -\frac{m\,v_0}{c}\ e^{\frac{-c\,t}{m}}\Big]_0^t$$

$$x = -\frac{m\,v_0}{c}\ e^{\frac{-c\,t}{m}} + \frac{m\,v_0}{c}\ e^{\frac{-c(o)}{m}}$$

$$x = \frac{m\,v_0}{c}\left(1 - e^{\frac{-c\,t}{m}}\right) \quad \ldots\ldots.(6)$$ ***Displacement as a function of*** $(\boldsymbol{t})$

Example:

If $F = -cv$ find the velocity and time equations as a function of displacement for a particle with initial velocity v_0 .

Solution:

$F = -cv$ and $F = mv\frac{dv}{dx}$

$$-cv = mv\frac{dv}{dx}$$

$$-\frac{c}{m}\int_0^x dx = \int_{v_0}^{v} dv$$

$$-\frac{c}{m}x = v - v_0$$

$$v = v_0 - \frac{c}{m}x \quad \ldots..(7)$$ ***Velocity as a function of*** $(\boldsymbol{x})$

[The speed of the body varies linearly with the displacement (distance)]

سرعة الجسم تتغير خطياً مع الازاحة

$$\therefore v = \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = v_0 - \frac{c}{m}x$$

$$\int_0^t dt = \int_0^x \frac{dx}{v_0 - \frac{c}{m}x} * \frac{-\frac{c}{m}}{-\frac{c}{m}}$$

$$v = v_0 - \frac{c}{m}x$$

$$\therefore dv = -\frac{c}{m}dx$$

$$\int \frac{dv}{v} = \ln v$$

$$\therefore \int \frac{dv}{v} = \int \frac{-\frac{c}{m}dx}{v_0 - \frac{c}{m}x} = \ln\left[v_0 - \left(\frac{c}{m}\right)x\right]$$

$$t = \int_0^x \frac{-\frac{c}{m}dx}{-\frac{c}{m}[v_0 - \left(\frac{c}{m}\right)x]}$$

$$t = -\frac{m}{c}\ln\left(v_0 - \left(\frac{c}{m}\right)x\right)\Big]_0^x$$

$$t = \frac{-m}{c}\left[\ln\left(v_0 - \left(\frac{c}{m}\right)x\right) - \ln v_0\right]$$

$$t = -\frac{m}{c}\ln\left[\frac{v_0 - \left(\frac{c}{m}\right)x}{v_0}\right] \qquad \text{(time as a function of } (x))$$

2.9 The Force as a Function of Time Only

$$F(t) = m\frac{dv}{dt}$$

$$dv = \frac{F(t)}{m}dt$$

By integrating

$$v(t) = \int \frac{F(t)}{m}\, dt \dots\dots (1)$$

$$v(t) = \frac{dx}{dt}$$

$$dx = v(t)dt$$

$$\int dx = \int v(t)\, dt \quad \dots\dots (2)$$

Sub Eqn. (1) in (2)

$$x = \int [\int \frac{F(t)}{m} dt]\, dt \dots\dots (3)$$

Example:

A block is initially at rest on a smooth horizontal surface. At time ($t = 0$) a constant increasing horizontal force is applied $F = ct$. Find the velocity and displacement as a function of time.

Solution:

$$F = ct = m\frac{dv}{dt}$$

$$dv = \frac{1}{m}ct\, dt$$

$$v = \frac{1}{m}\int_0^t ct\, dt$$

$$v = \frac{ct^2}{2m}$$

$\therefore v = \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{ct^2}{2m}$

$dx = \frac{ct^2}{2m} dt$

$x = \int_0^t \frac{ct^2}{2m} dt$

$x = \frac{ct^3}{6m}$

2.10 Vertical Motion in a Resisting Medium Terminal Velocity

Linear Resistance

An object falling vertically through the air or through any fluid is subject to ***viscous resistance***. If the resistance is ***proportional*** to the ***first power*** of (v), we can express this force as $(-c\vec{v})$ regardless of the sign of (v) because the resistance is always ***opposite*** to the direction of motion. The constant of proportionality c depends on the ***size and shape of the object*** and the ***viscosity of the fluid***.

يتعرض الجسم الذي يسقط رأسياً عبر الهواء أو من خلال أي مائع لمقاومة اللزوجة. إذا كانت المقاومة تتناسب مع القوة الأولى لـ (v) (الحالة الخطية)، فيمكننا التعبير عن هذه القوة كـ $(-c\vec{v})$ بغض النظر عن اشارة (v)لأن المقاومة تكون دائمًا معاكسة لاتجاه الحركة. ثابت التناسب c يعتمد على حجم وشكل الجسم ولزوجة المائع.

Let us take the *x* axis to be ***positive*** upward. The differential equation of motion is then

$-mg - c\vec{v} = m\frac{d\vec{v}}{dt}$ ……(1) ***Linear Equation***

$\int_0^t dt = \int_{v_0}^{v} \frac{mdv}{-mg-cv}$

$\frac{t}{m} = \int_{v_0}^{v} \frac{dv}{-mg-cv} = -\int_{v_0}^{v} \frac{dv}{(mg+cv)}$

$\frac{t}{m} = -\int_{v_0}^{v} \frac{dv}{mg+cv}$

$\frac{t}{m} = -\frac{1}{c_1} \ln(mg+cv)\Big]_{v_0}^{v}$

$\frac{t}{m} = -\frac{1}{c_1} [\ln(mg+cv) - \ln(mg+v_0)]$

$$\therefore t = -\frac{m}{c}\ln\left(\frac{mg+cv}{mg+cv_0}\right) \quad \ldots\ldots(2) \qquad \textit{Equation of Time}$$

Eqn. (2) represents time in term of velocity; by solve Eqn. (2)

$$e^{\frac{-ct}{m}} = \frac{mg+cv}{mg+cv_0}$$

$$mg + cv = (mg + cv_0)e^{\frac{-ct}{m}}$$

$$cv = -mg + mg\, e^{\frac{-ct}{m}} + cv_0\, e^{\frac{-ct}{m}} \quad] \div c$$

$$\therefore v = -\frac{mg}{c} + \frac{mg}{c}\, e^{\frac{-ct}{m}} + v_0\, e^{\frac{-ct}{m}}$$

$$\therefore v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{\frac{-ct}{m}} \ldots\ldots(3) \quad \textit{Velocity as function of time}$$

Eqn. (3) represents velocity in term of time.

When $\left(t \gg \frac{m}{c}\right)$ $then\ e^{\frac{-ct}{m}} = 0$ so, the exponential term can be neglected

$$\therefore v = -\frac{mg}{c} \quad \ldots\ldots(4) \qquad \textbf{\textit{Terminal velocity}}$$

***Terminal velocity*:** It is that velocity at which the force resistance is just equal and opposite to the weight of the body so that the total force on the body is zero and so the acceleration is zero.

سرعة المنتهى: هي تلك السرعة التي تكون فيها مقاومة القوة مساوية تماما ومعاكسة لوزن الجسم بحيث تكون القوة الكلية على الجسم تساوي الصفر وبالتالي فإن التعجيل سيكون صفر.

$\frac{mg}{c} = v_t$	***Terminal speed***	سرعة المنتهى
$\tau = \frac{m}{c}$	***Characteristic time***	الزمن النوعي

Then Eqn. (3) becomes:

$v = -v_t + (v_t + v_0)e^{-t/\tau}$ ……(4) ***Velocity in term of terminal velocity***

These two terms represent two velocities; the terminal velocity v_t which exponentially (fades in) and the initial velocity v_0 which exponentially (fades out) due to the action of the viscous drag force.

هذان الحدان يمثلان سرعتين. سرعة المنتهى v_t التي تتلاشى أسياً والسرعة الأبتدائية v_0 التي تتلاشى أسياً بسبب تأثير قوة اللزوجة.

We can find displacement by integrate Eqn. (3)

$$\therefore \frac{dx}{dt} = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{\frac{-ct}{m}}$$

$$\int_{x_0}^{x} dx = -\frac{mg}{c}\int_0^t dt + \frac{mg}{c}\int_0^t e^{\frac{-ct}{m}}\,dt + \int_0^t v_0\, e^{\frac{-ct}{m}}\,dt$$

$$x - x_0 = -\frac{mg}{c}t + \frac{mg}{c}\left(\frac{-c/m}{-c/m}\right)\int_0^t e^{\frac{-ct}{m}}dt + \frac{v_0(-c/m)}{(-c/m)}\int_0^t e^{\frac{-ct}{m}}\,dt$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\int_0^t \frac{-c}{m} e^{\frac{-ct}{m}}dt \int_0^t -\frac{v_0 m}{c}\frac{-c}{m} e^{\frac{-ct}{m}}dt$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\left[e^{\frac{-ct}{m}}\right]_0^t - \frac{v_0 m}{c}\left[e^{\frac{-ct}{m}}\right]_0^t$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\left[e^{\frac{-ct}{m}} - e^0\right] - \frac{v_0 m}{c}\left[e^{\frac{-ct}{m}} - e^0\right]$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\left(e^{\frac{-ct}{m}} - 1\right) - \frac{v_0 m}{c}\left(e^{\frac{-ct}{m}} - 1\right)$$

$$x - x_0 = -\frac{mg}{c}t + \frac{m^2 g}{c^2}\left(1 - e^{\frac{-ct}{m}}\right) + \frac{v_0 m}{c}\left(1 - e^{\frac{-ct}{m}}\right)$$

$$x - x_0 = -\frac{mg}{c}t + \left(\frac{m^2 g}{c^2} + \frac{m v_0}{c}\right)\left(1 - e^{\frac{-ct}{m}}\right)$$ …….(5) ***Displacement Equation***

Also, write in the form

$$x = x_0 - v_t t + X_1\left(1 - e^{\frac{-ct}{m}}\right) \quad \dots\dots(6)$$

where $X_1 = \frac{m^2 g}{c^2} + \frac{m v_0}{c} = g\tau^2 + v_0\tau$

In particular, for an object dropped from ***rest*** $v_0 = 0$,

From Eqn. (4)

$$v = -v_t + (v_t + v_0)e^{-t/\tau} = -v_t + v_t\, e^{-t/\tau}$$

- When $(t = \tau)$

  $v = \left(1 - e^{-t/\tau}\right)v_t$

  $v = (1 - e^{-1})v_t$

$\therefore When\ (t = \tau) \rightarrow v = (1 - e^{-1})vt$

- When $t = 2\tau$

  $v = \left(1 - e^{-2\tau/\tau}\right)v_t$

  $v = (1 - e^{-2})v_t$

$\therefore when\ (t = 2\tau) \rightarrow v = (1 - e^{-2})v_t$

Thus, after *one characteristic time* the speed is $(1 - e^{-1})$ *times* the terminal speed, after *two characteristic times* it is the factor $(1 - e^{-2})$ of and so on. After an interval of the speed is within 1% of the terminal value, namely, $(1 - e^{-5})v_t = 0.99995\ v_t$

وبالتالي اذا اسقط جسم من السكون فبعد زمن نوعي واحد، تكون السرعة $(1 - e^{-1})$ أضعاف سرعة المنتهى ، وبعد ضعفين للزمن النوعي تكون سرعته $(1 - e^{-2})$هكذا. بعد فترة زمنية تصل سرعة الجسم في حدود 1 ٪ من قيمة سرعة المنتهى ، أي $(1 - e^{-5})v_t = 0.99995\ v_t$